Mental exercise

Exponents

PART 1

$$y = 2^x$$

$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{1}$	$\frac{2}{1}$	$\frac{4}{1}$	$\frac{8}{1}$	$\frac{16}{1}$	$\frac{32}{1}$	$\frac{64}{1}$
2^{-5}	2^{-4}	2^{-3}	2^{-2}	2^{-1}	2^{0}	2^{1}	2^{2}	2^{3}	2^{4}	2^{5}	2^{6}

$$y = \pi^x$$

$$\frac{1}{\pi^5} \quad \frac{1}{\pi^4} \quad \frac{1}{\pi^3} \quad \frac{1}{\pi^2} \quad \frac{1}{\pi} \quad \frac{1}{1} \quad \frac{\pi}{1} \quad \frac{\pi^2}{1} \quad \frac{\pi^3}{1} \quad \frac{\pi^4}{1} \quad \frac{\pi^5}{1} \quad \frac{\pi^6}{1}$$

$$y = \left(\frac{\pi}{2}\right)^x$$

$$\frac{32}{\pi^5} \quad \frac{16}{\pi^4} \quad \frac{8}{\pi^3} \quad \frac{4}{\pi^2} \quad \frac{2}{\pi} \quad \frac{1}{1} \quad \frac{\pi}{2} \quad \frac{\pi^2}{4} \quad \frac{\pi^3}{8} \quad \frac{\pi^4}{16} \quad \frac{\pi^5}{32} \quad \frac{\pi^6}{64}$$

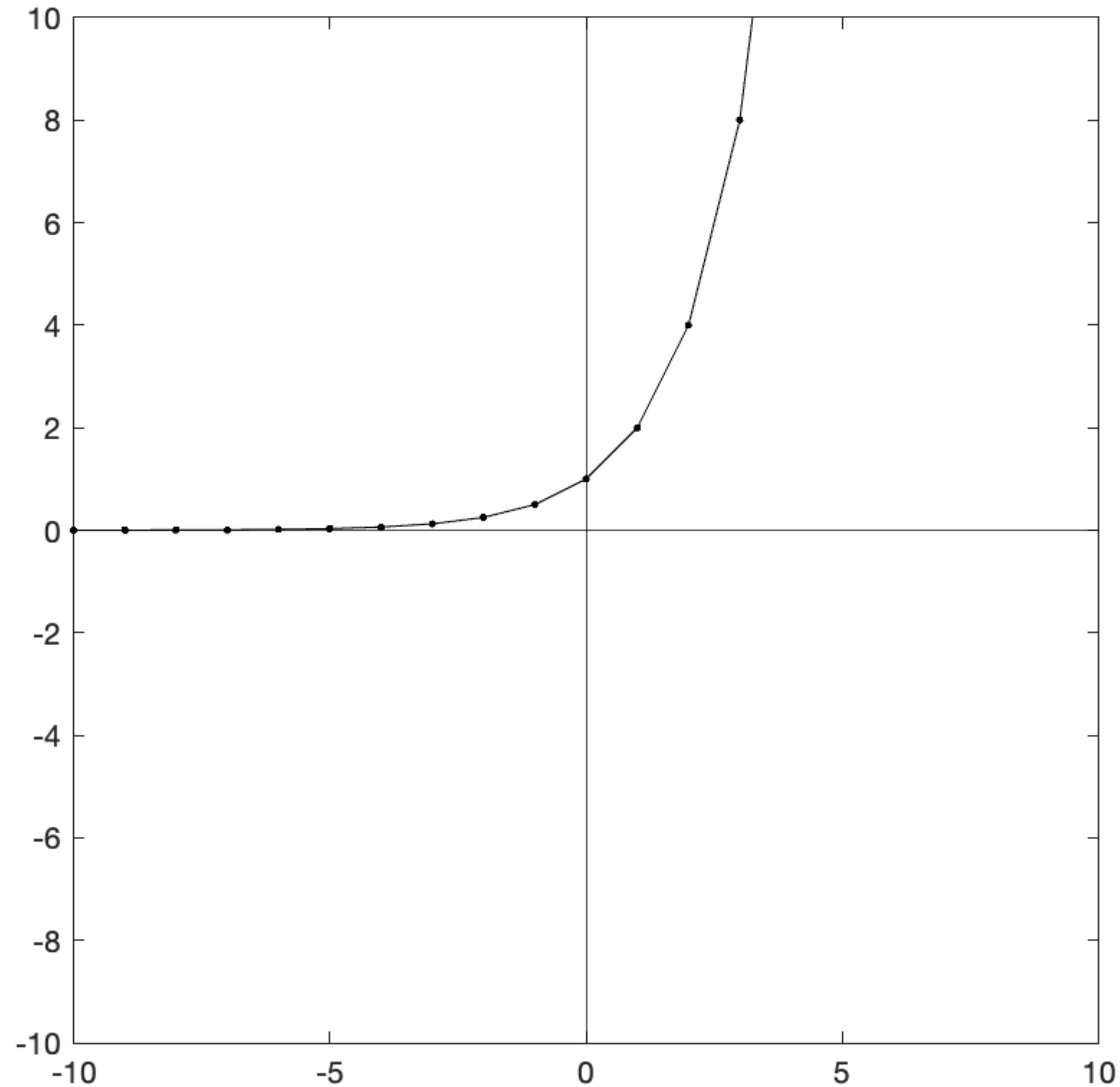
$$y = a^x$$

$$a > 0 \qquad a \in \mathbb{R}$$

a is called the "base"

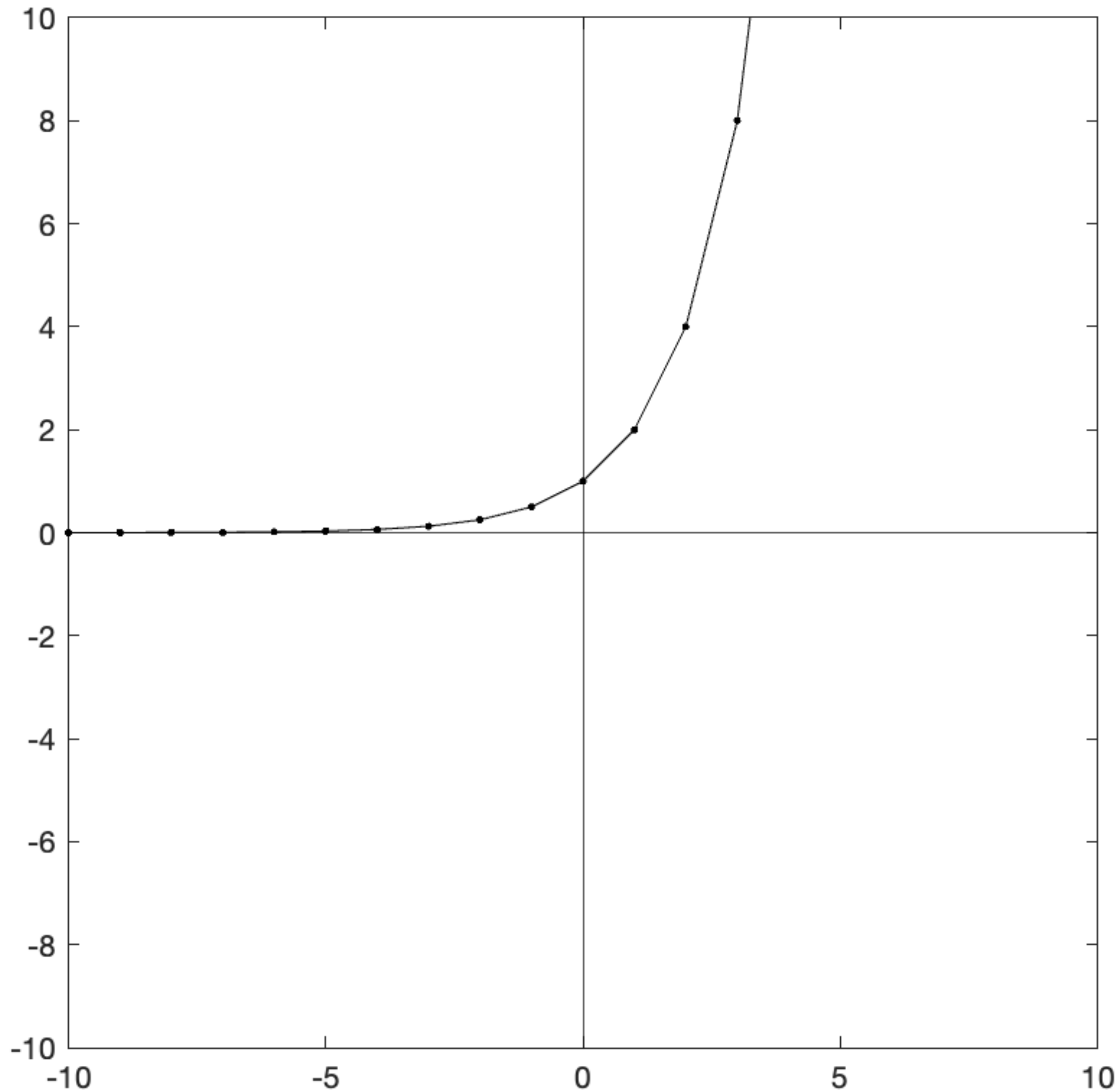
and y is an "exponential function"

Algebraic: 1, 2, 4, 8…

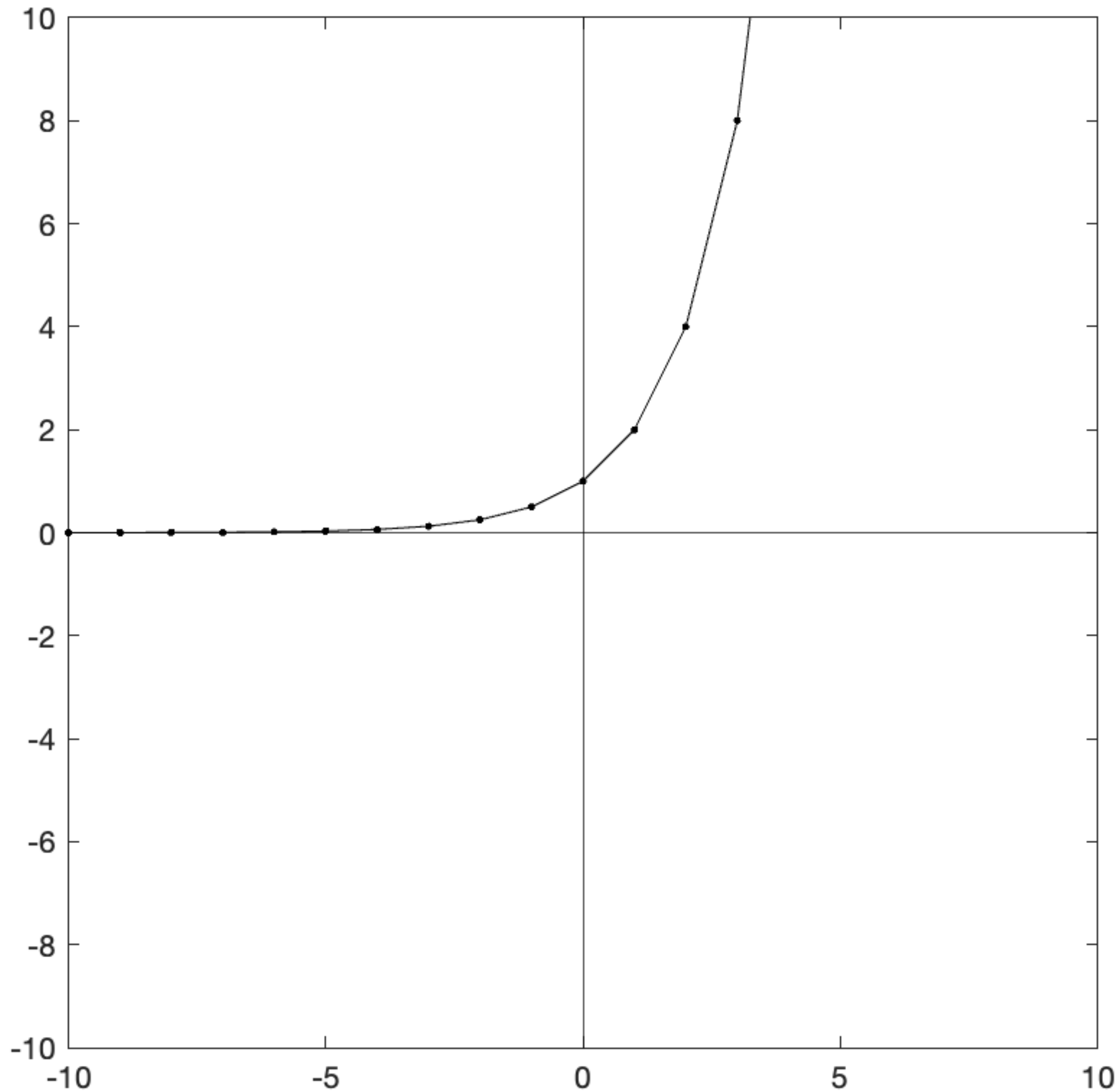


$$y = 2^x$$

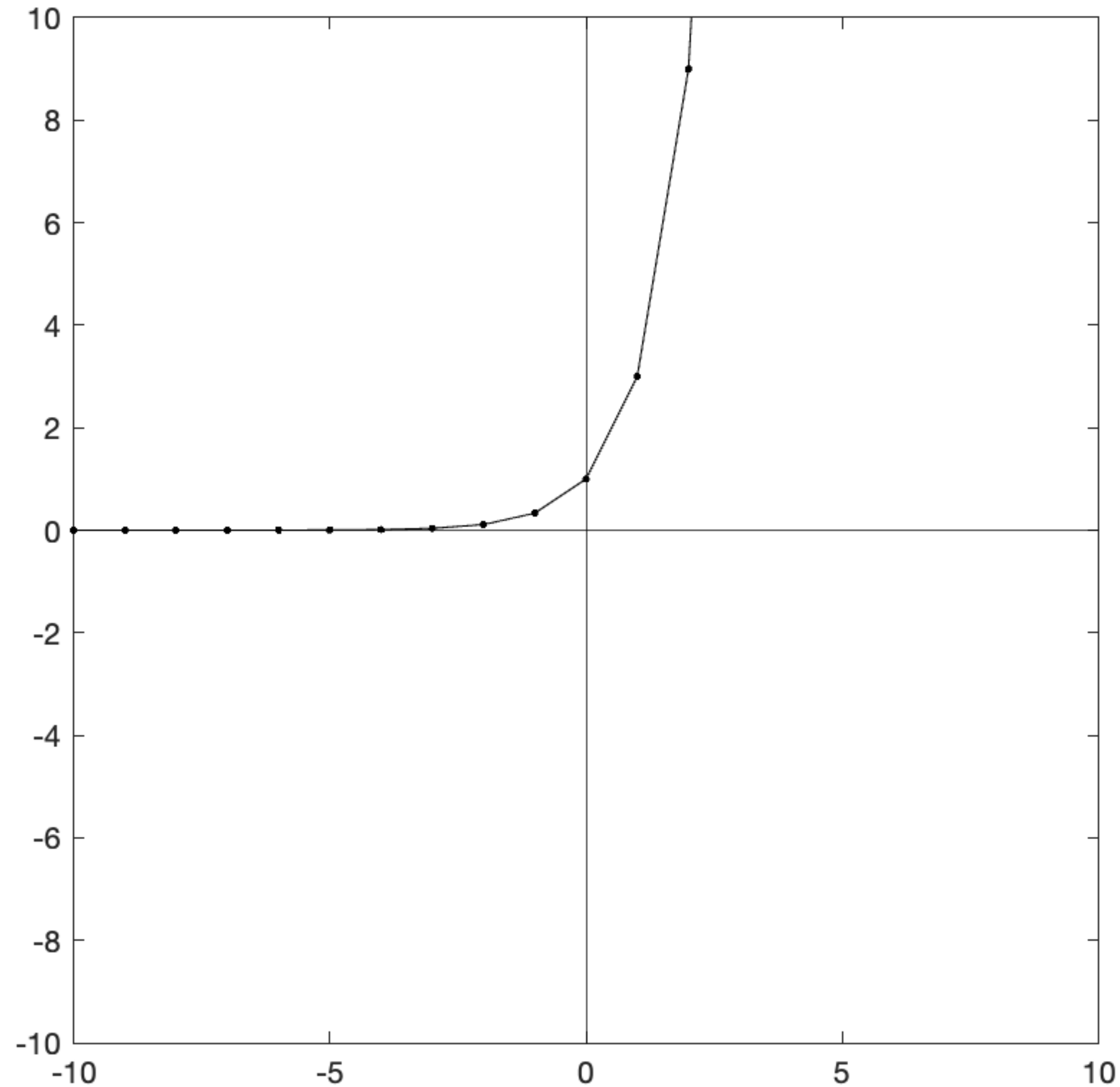
Maybe: a process repeated x times - each step doubles



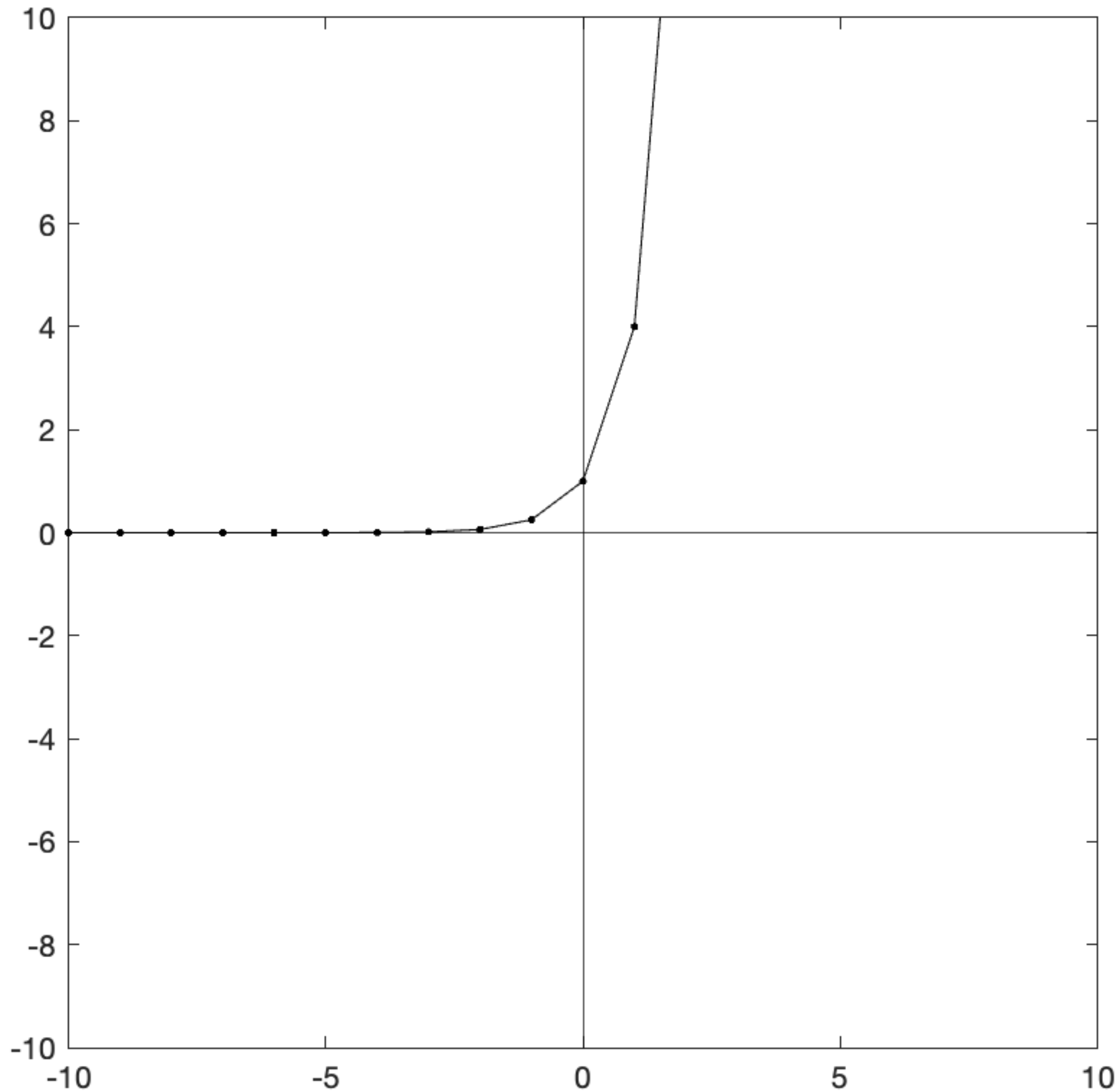
$$y = 2^x$$



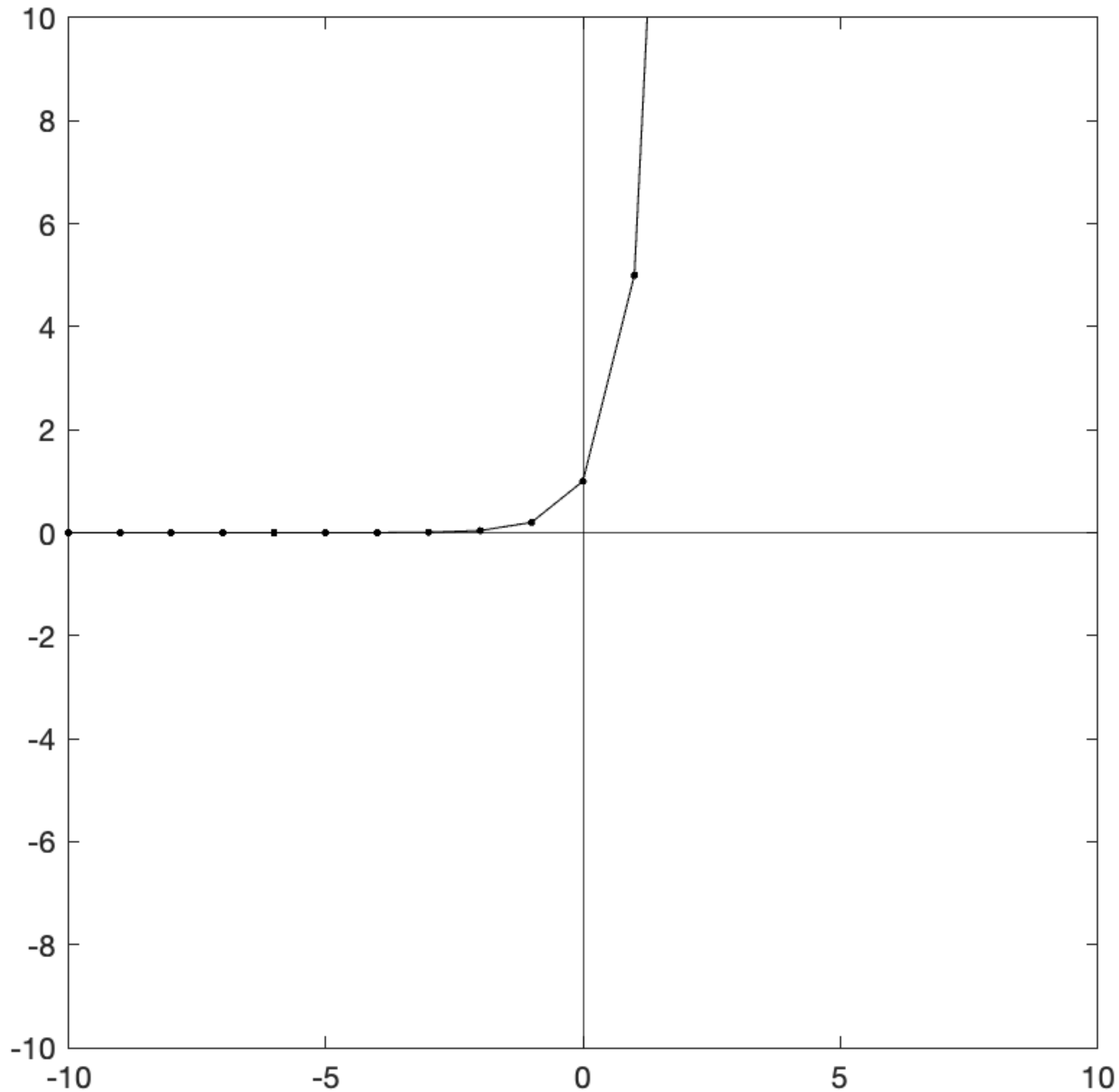
$$y = 2^x$$



$y = 3^x$



$$y = 4^x$$



$y = 5^x$

$x \in \mathbb{N}, \mathbb{Z}$

$y = 5^x$

$x \in \mathbb{Q}$ $\qquad x = \frac{p}{q}$

$x \in \mathbb{R}$

$\frac{1}{1}$ $\quad \frac{\sqrt{5}}{1}$ $\quad \frac{\sqrt[3]{5^2}}{1}$ $\quad \frac{5}{1}$ $\quad \frac{25}{1}$

5^0 $\quad 5^{\frac{1}{2}}$ $\quad 5^{\frac{2}{3}}$ $\quad 5^1$ $\quad 5^{\sqrt{2}}$ $\quad 5^2$ $\quad 5^{\pi}$

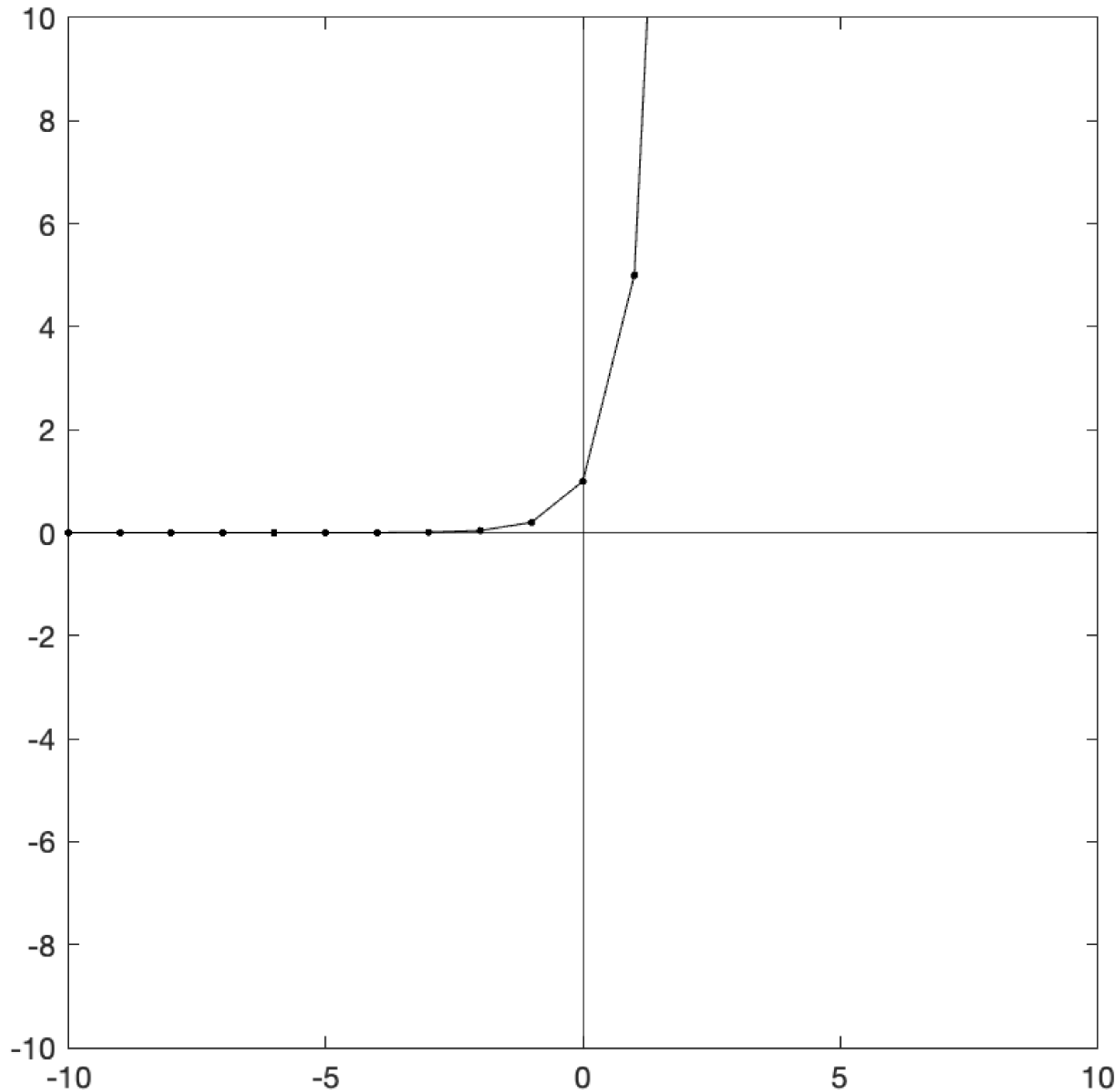
$5^{\frac{4}{3}} 5^{\frac{14}{10}} \quad 5^{\frac{29}{20}} 5^{\frac{3}{2}}$

So we may use any positive real as a base

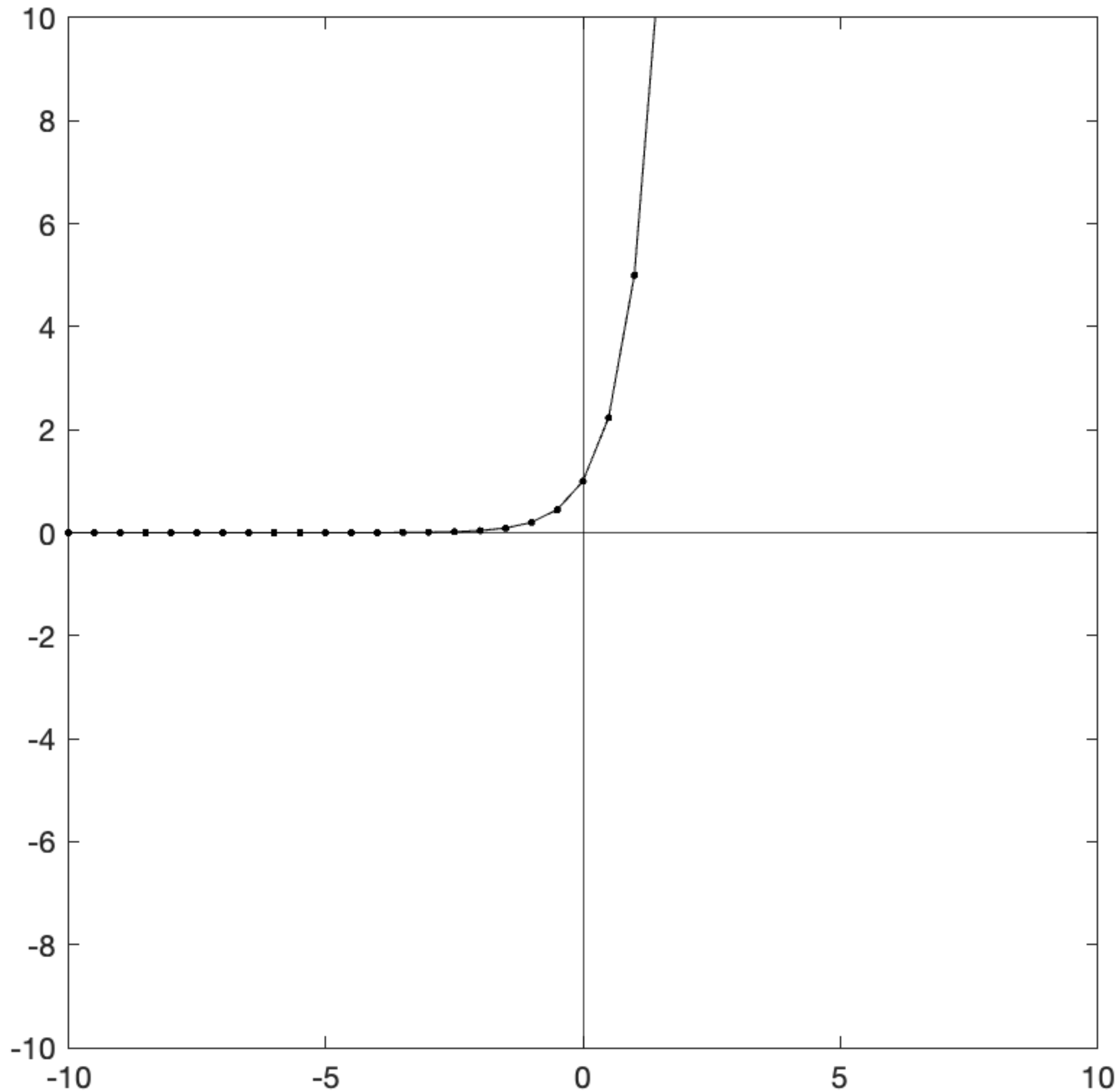
And it is continuous over all real x, positive and negative

$$y = a^x$$

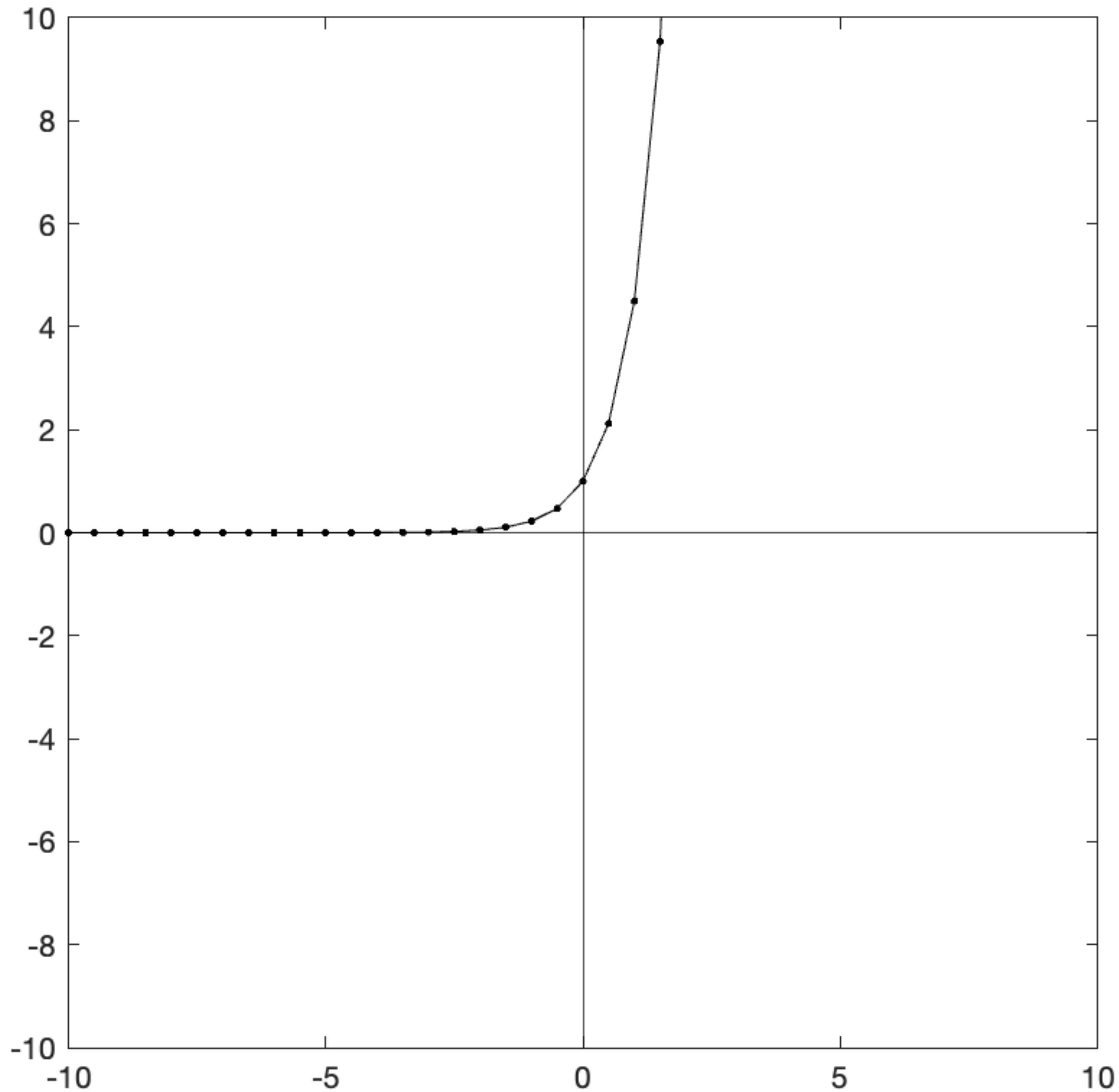
$$a > 0 \qquad a \in \mathbb{R} \qquad x \in \mathbb{R}$$



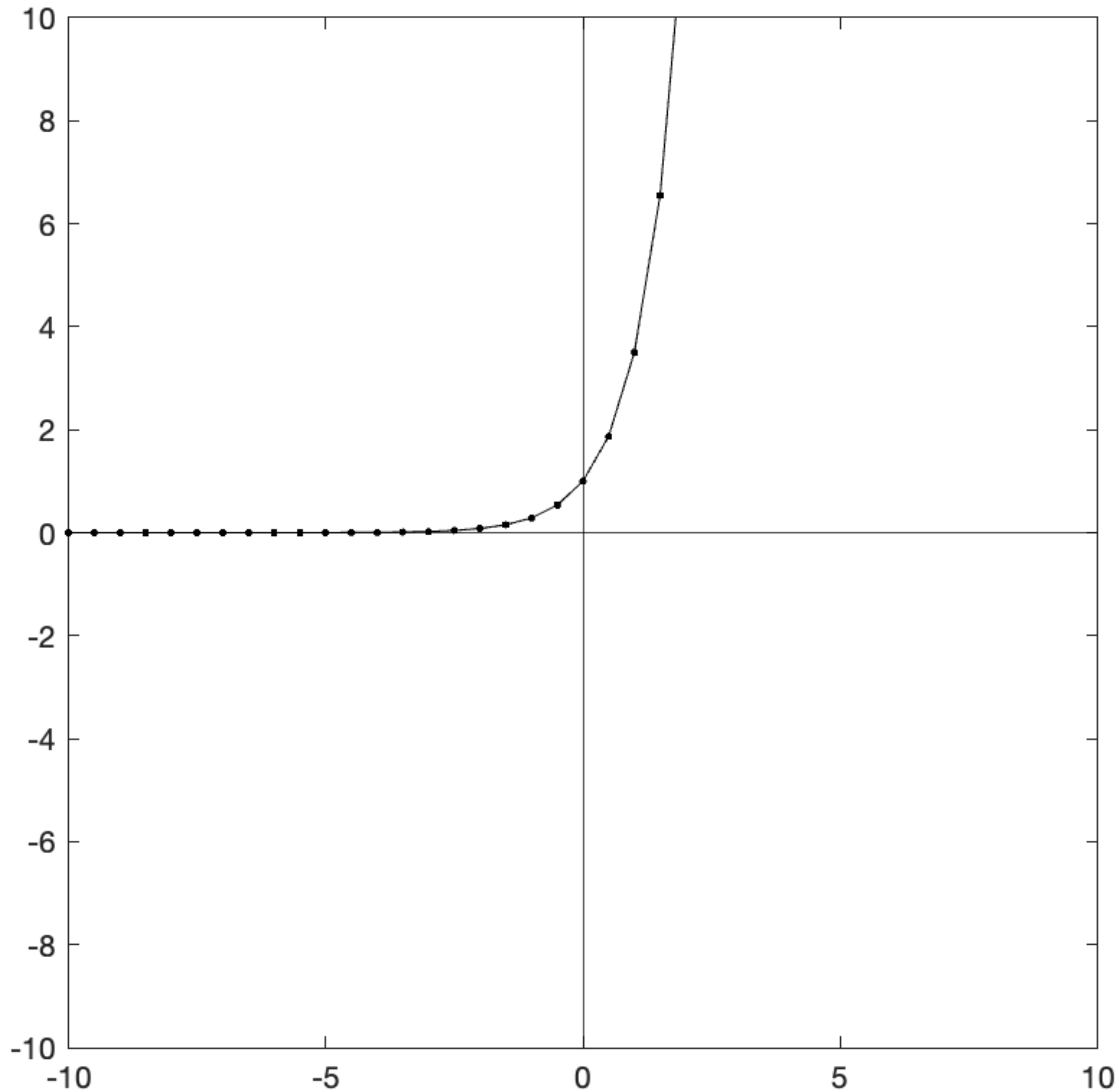
$$y = 5^x$$



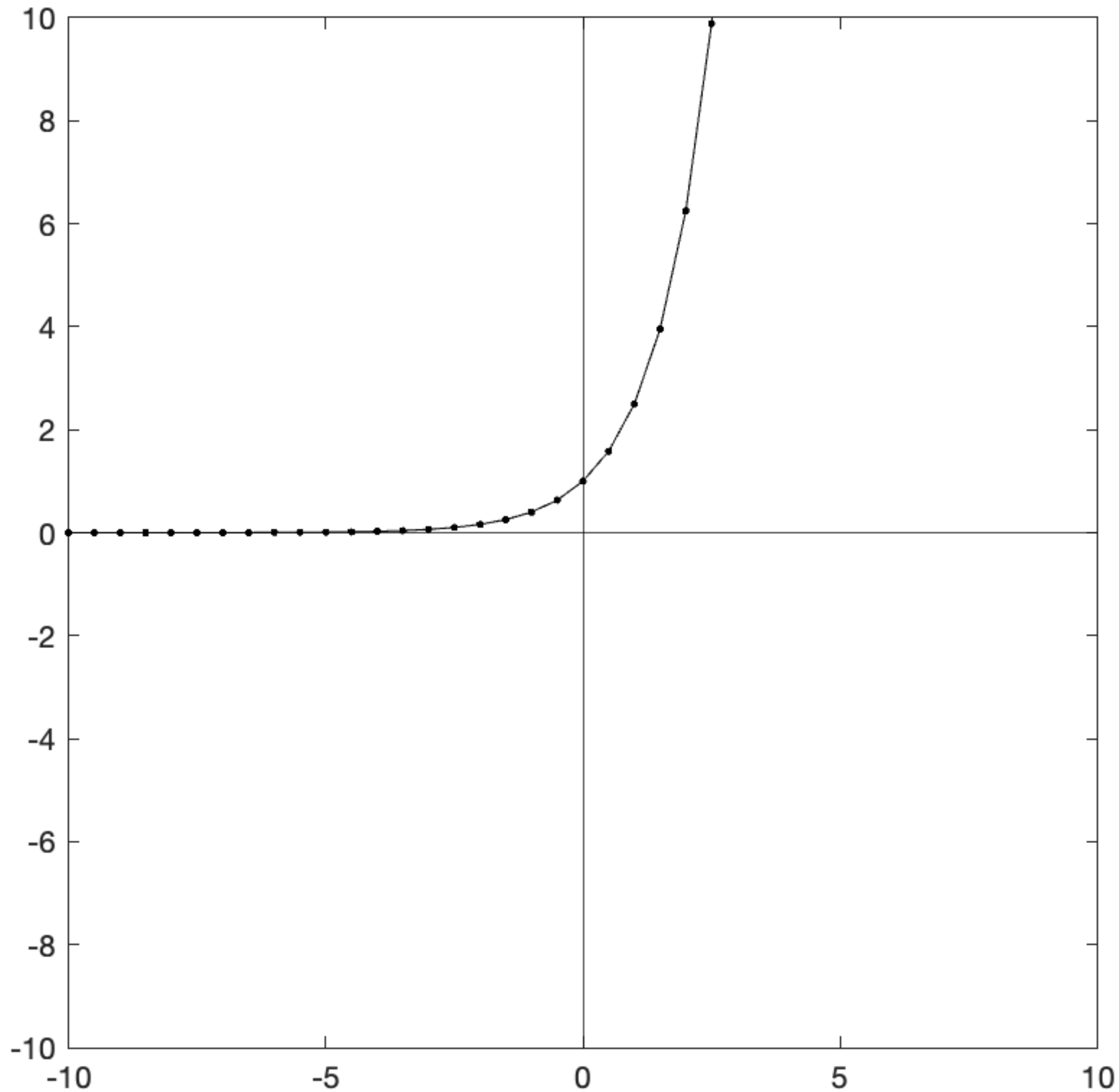
$y = 5^x$



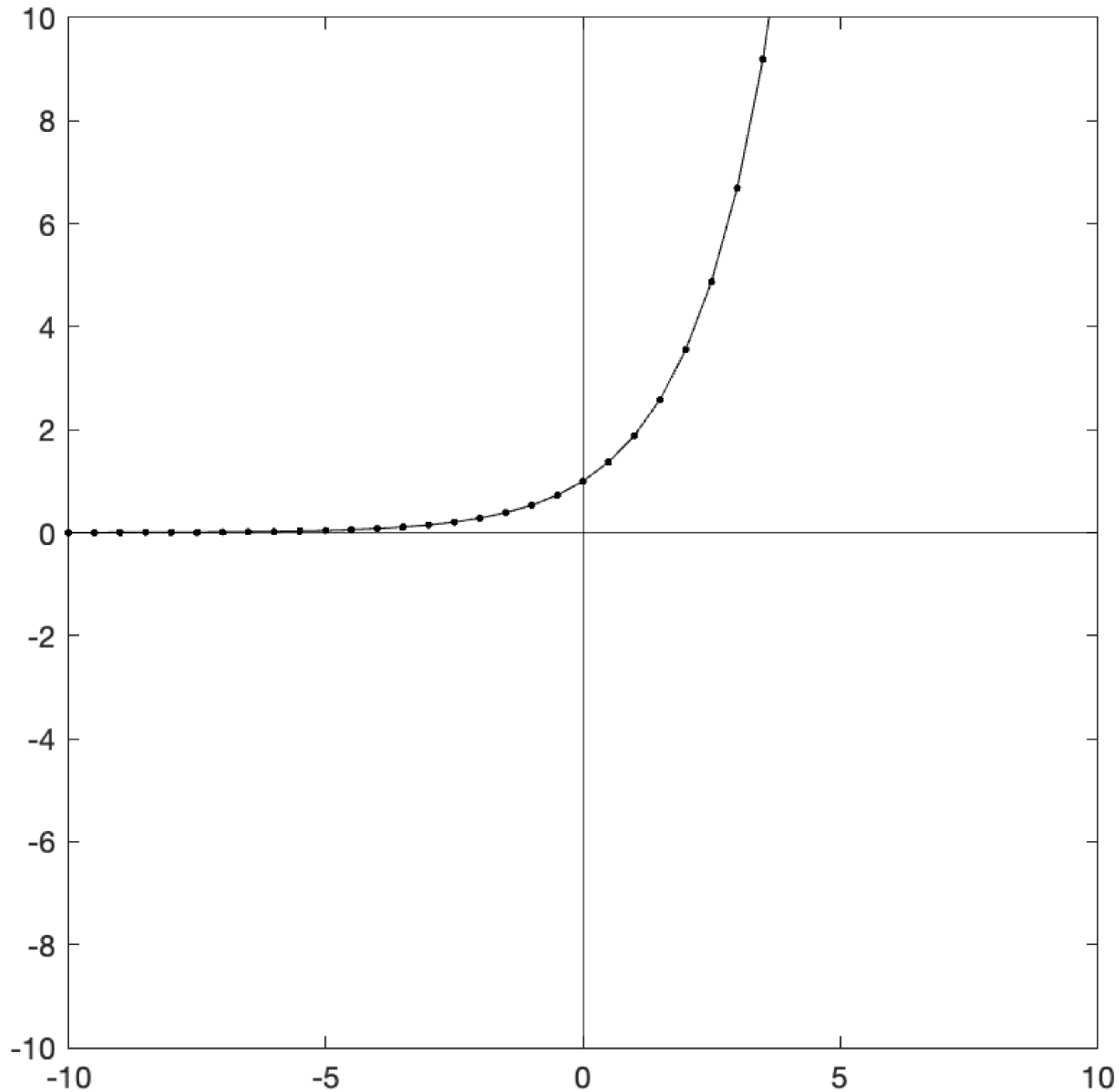
$$y = 4.5^x$$



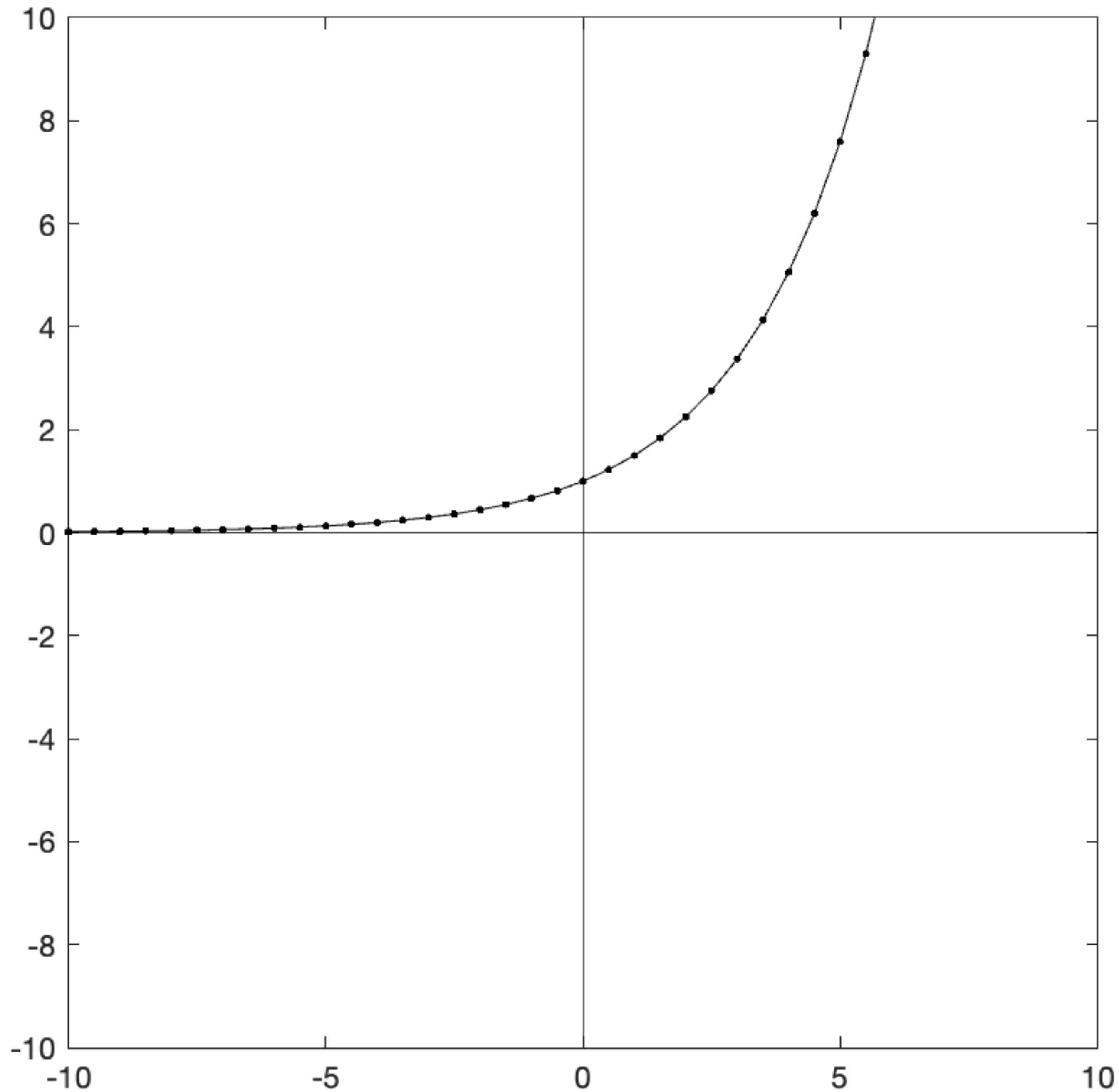
$$y = 3.5^x$$



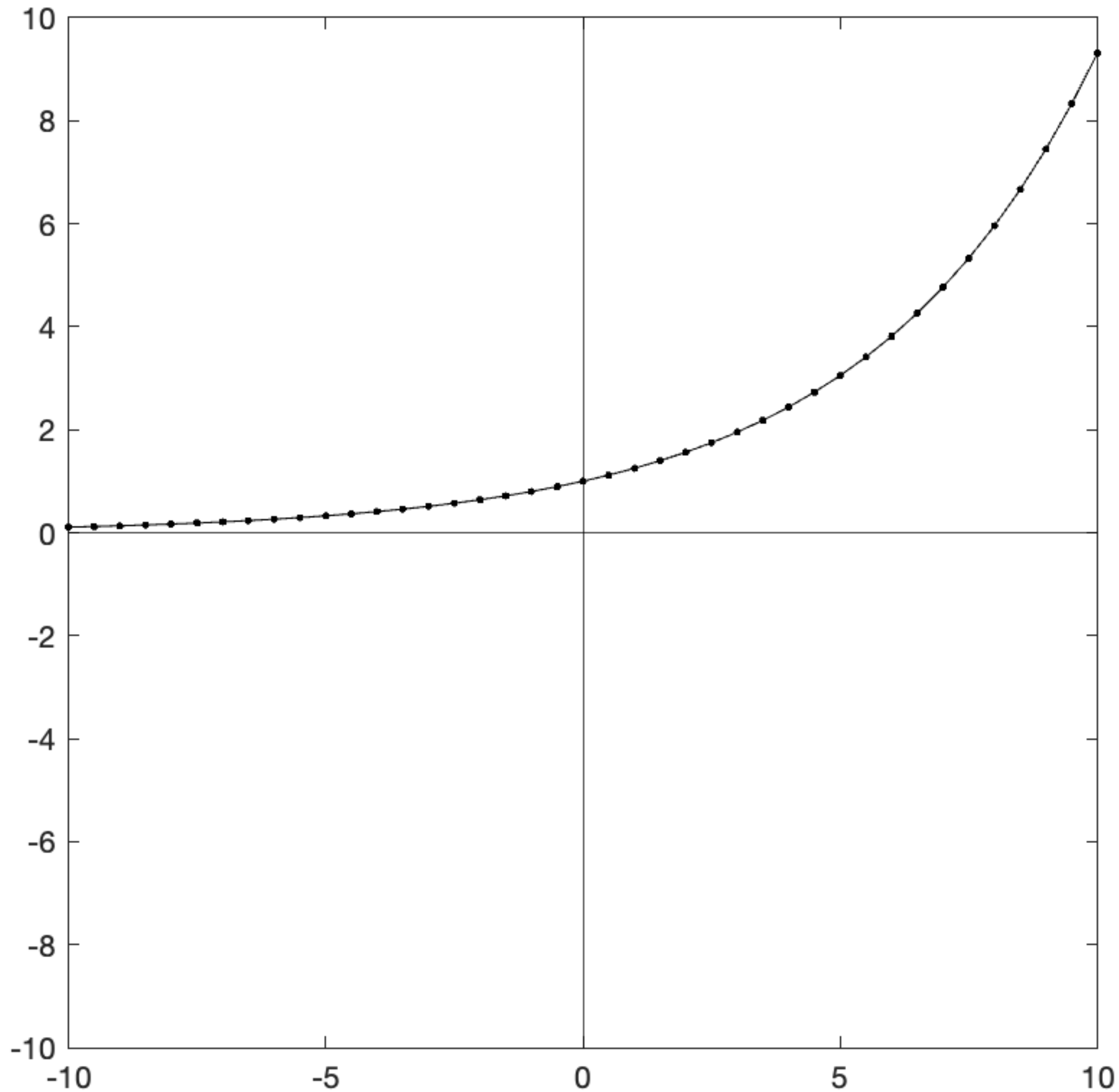
$$y = 2.5^x$$



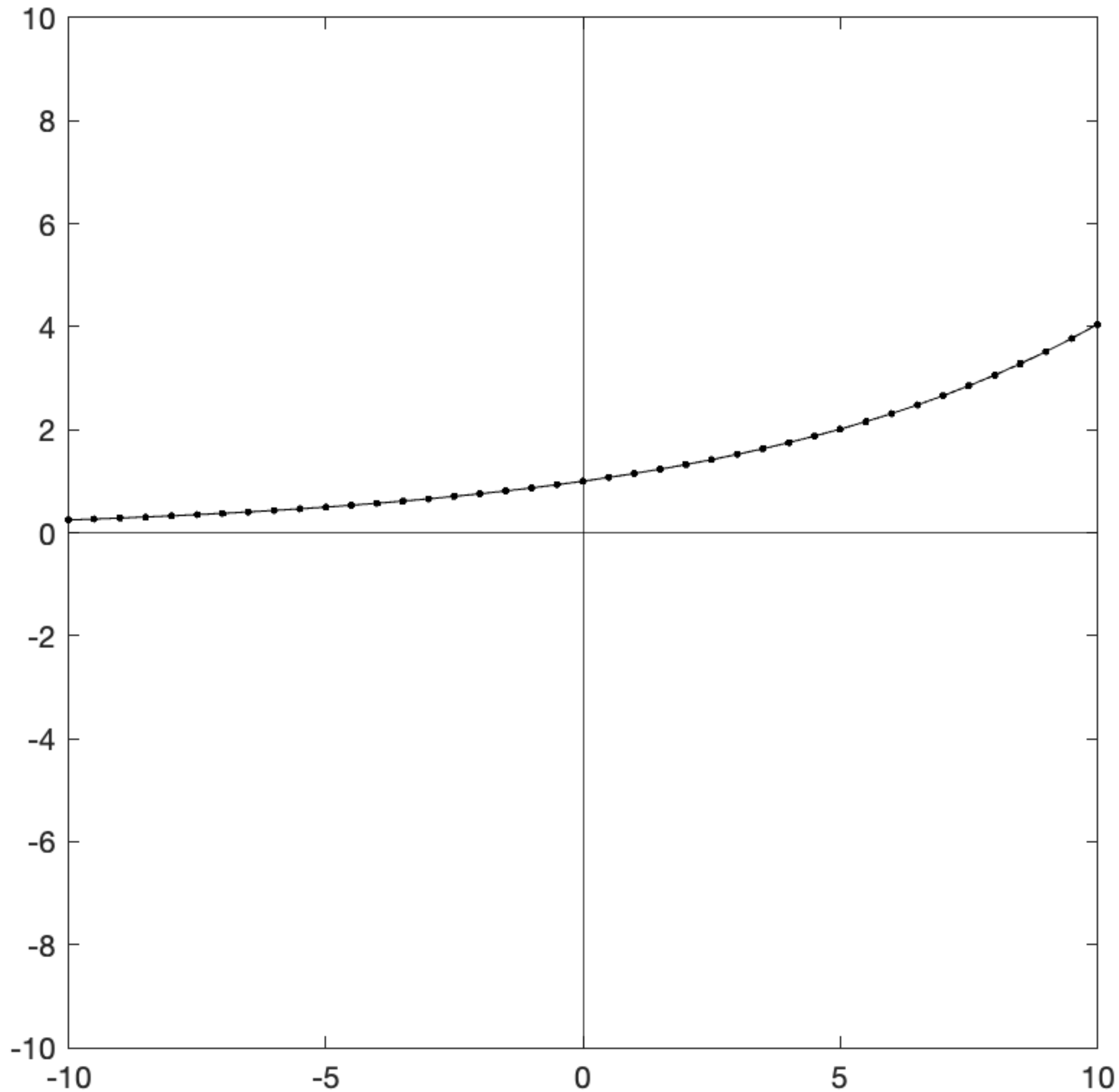
$$y = (0.6\pi)^x$$



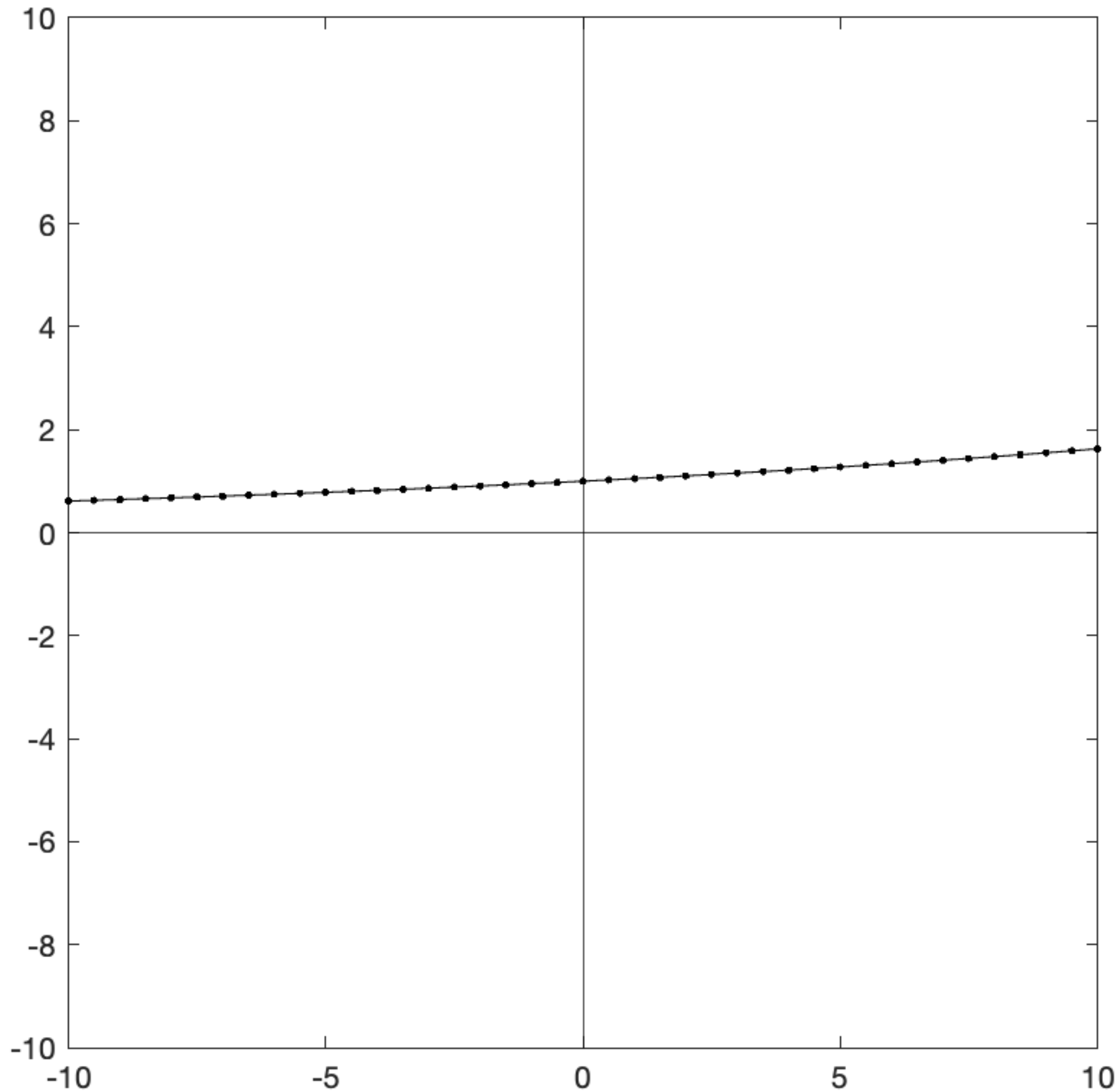
$y = 1.5^x$



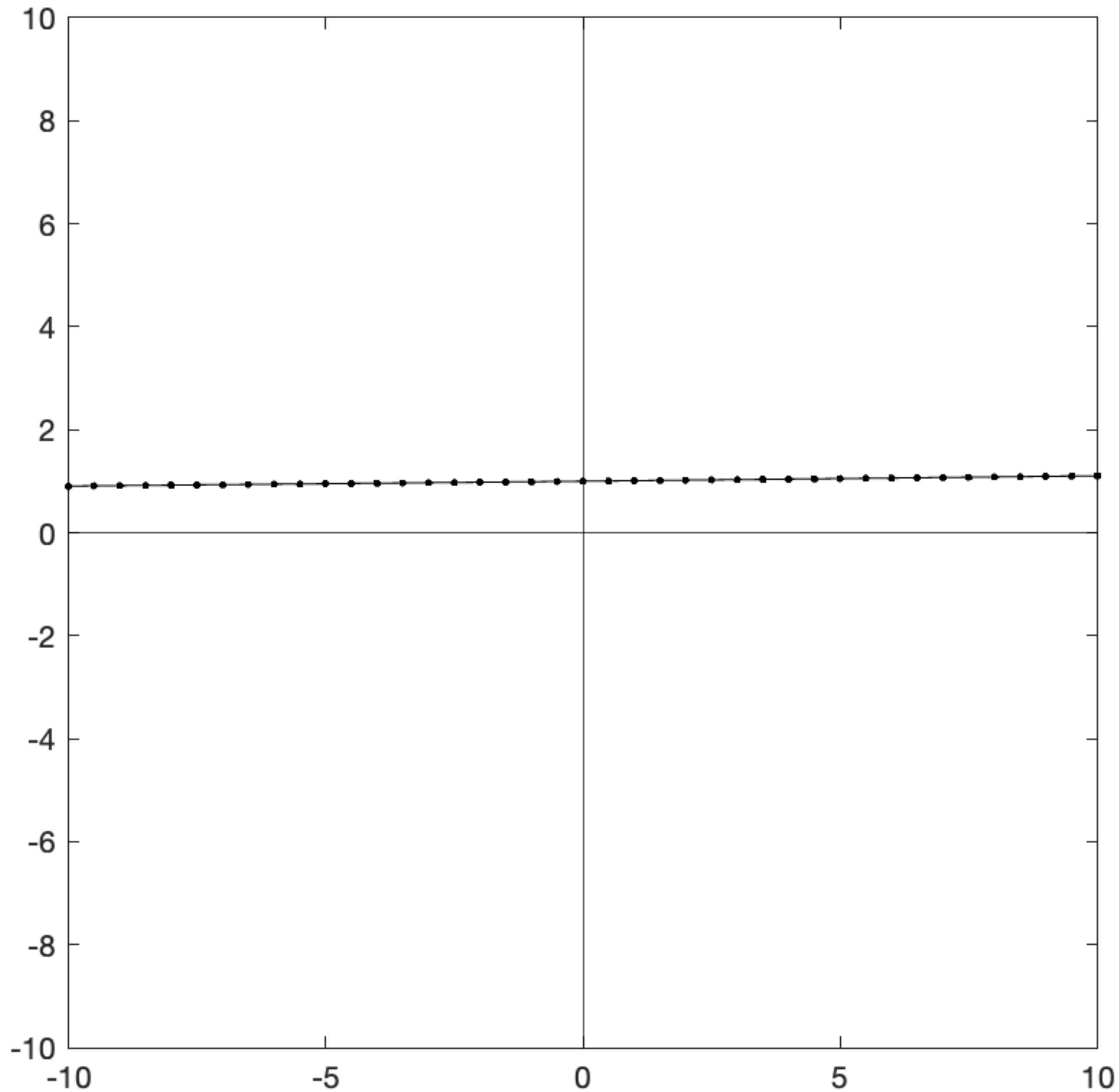
$$y = 1.25^x$$



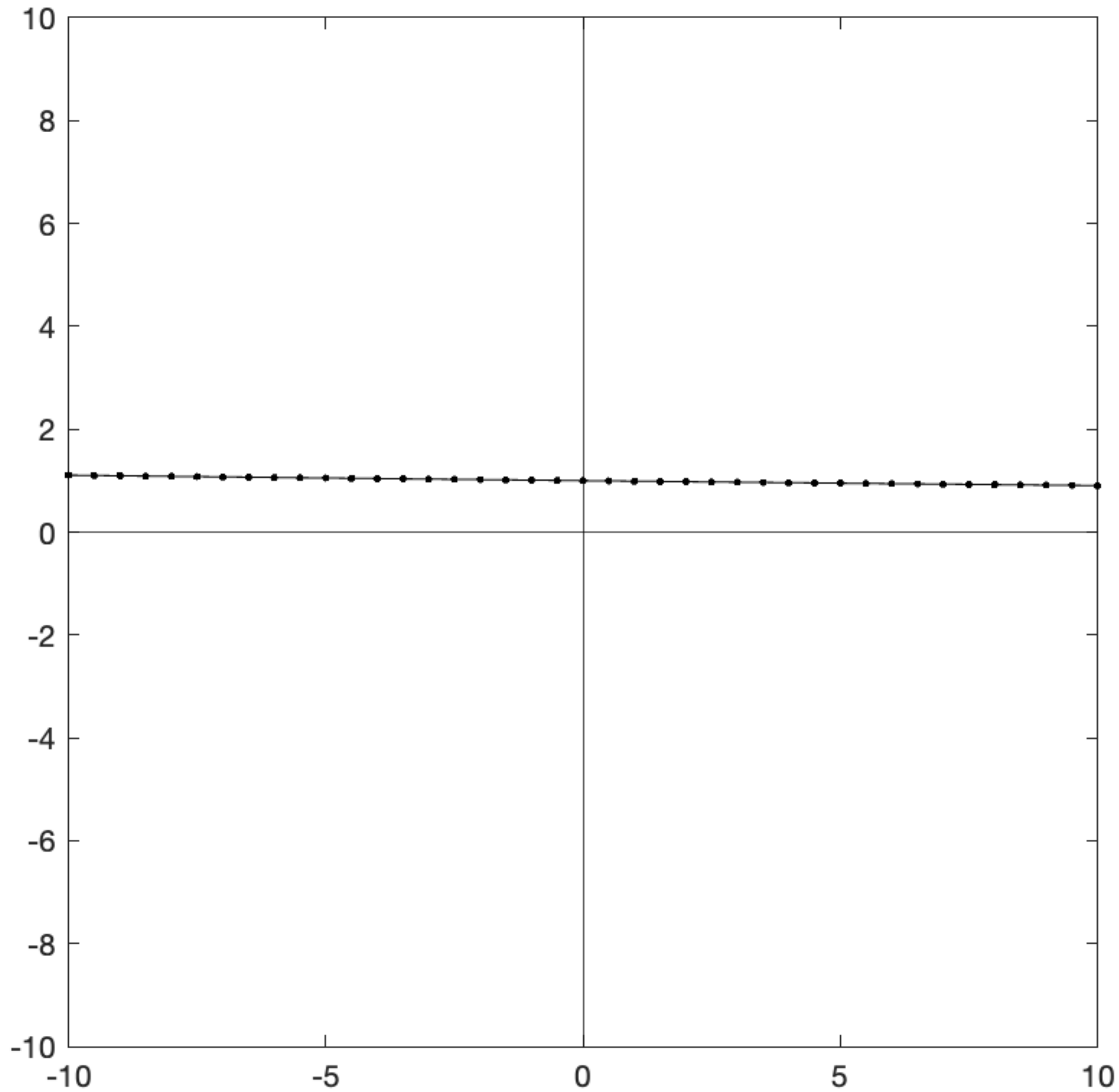
$$y = 1.15^x$$



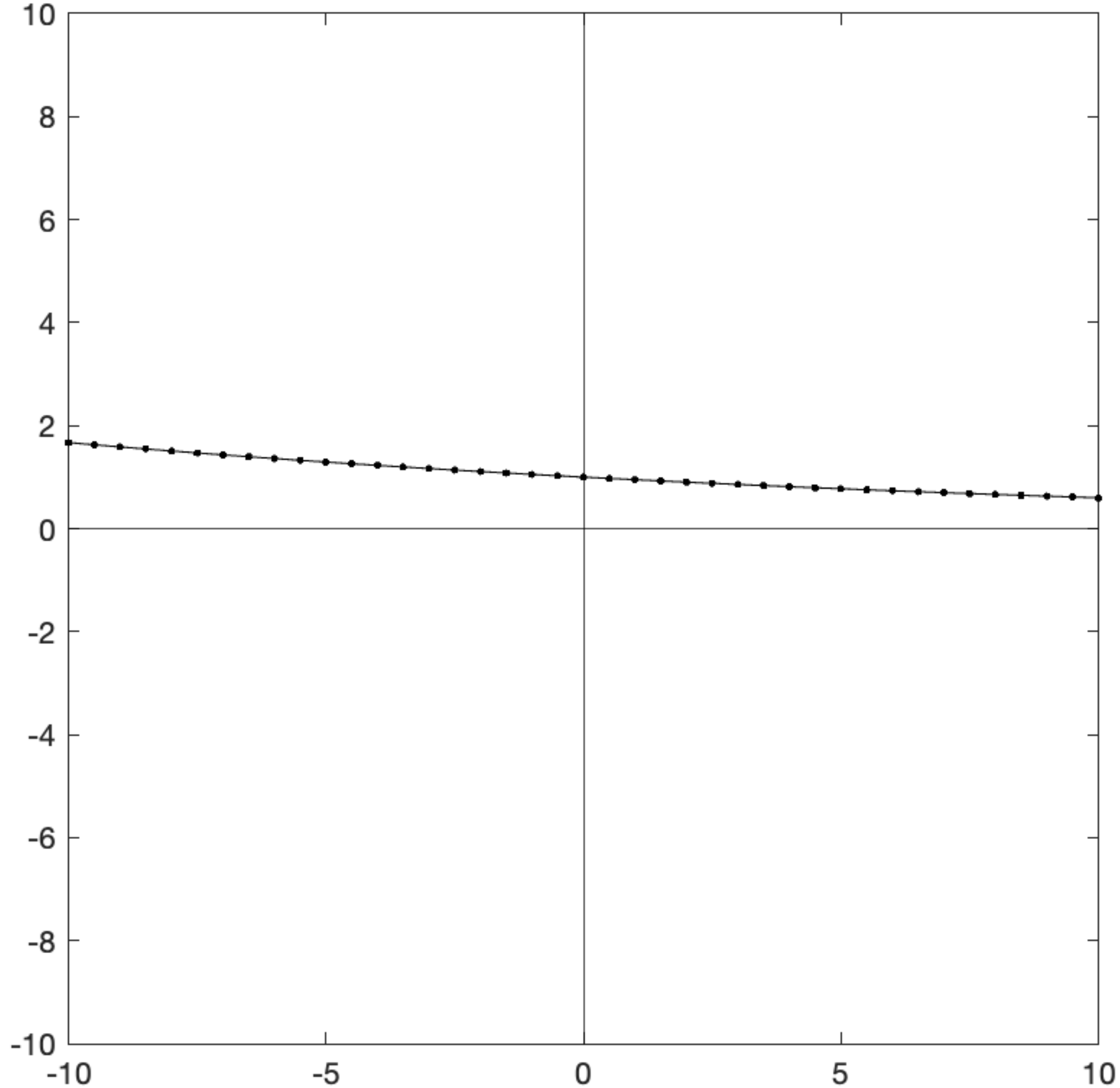
$$y = 1.05^x$$



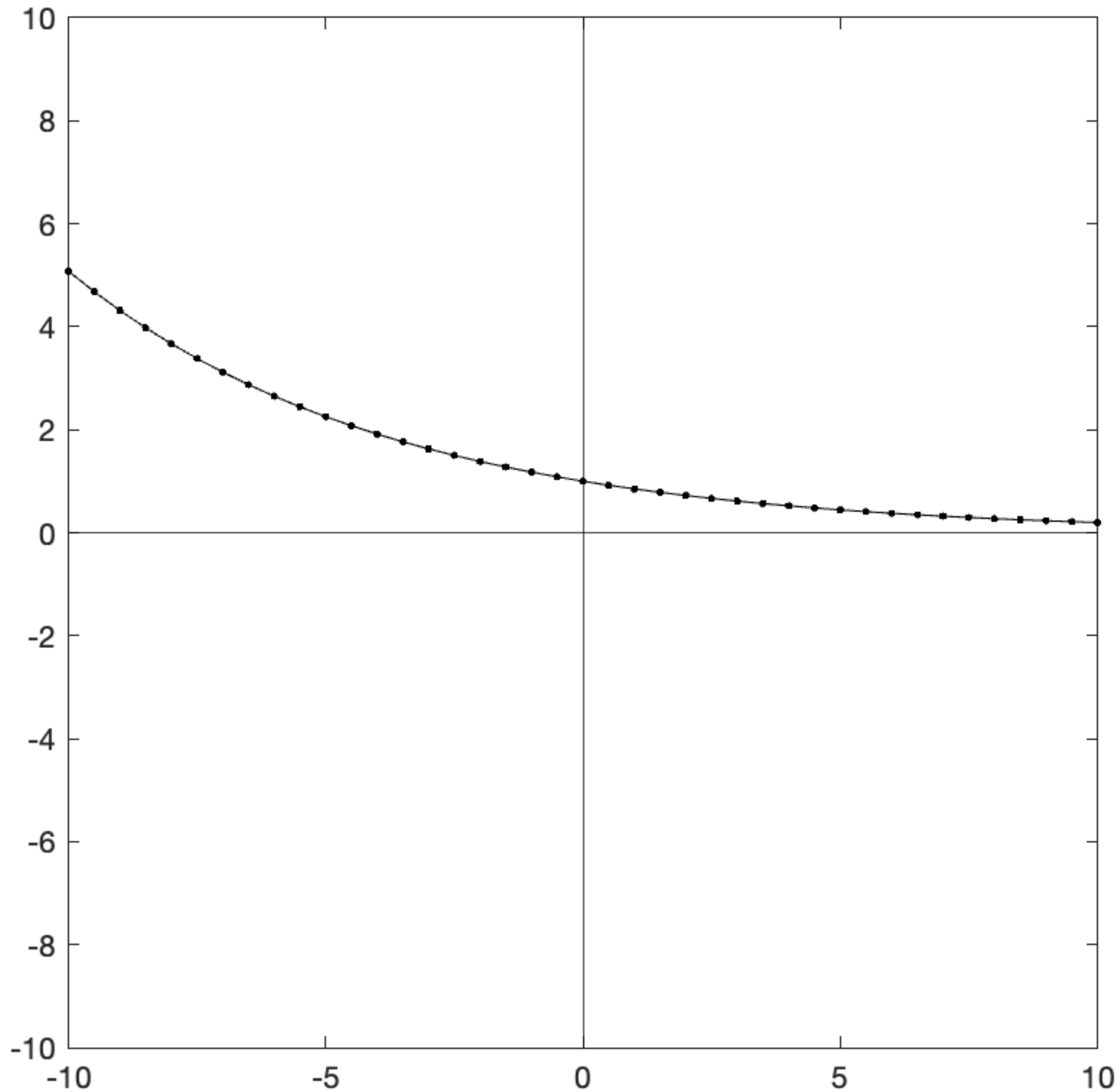
$$y = 1.01^x$$



$$y = 0.99^x$$



$$y = 0.95^x$$



$$y = 0.85^x$$

Can we have symmetric curves? When?

$$y_1 = a_1^x \qquad\qquad y_2 = a_2^x$$

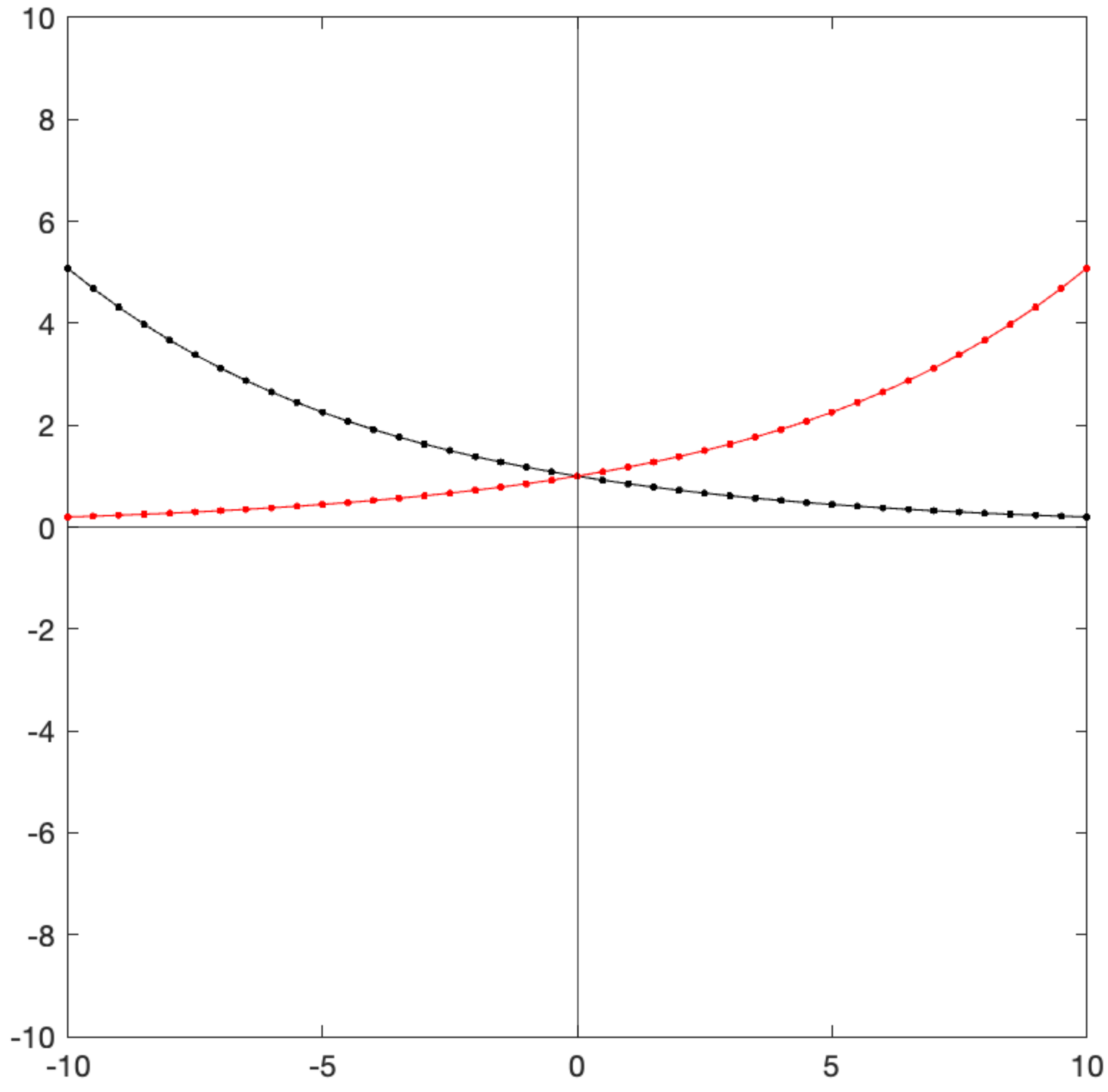
$$y_1(x) = y_2(-x)$$

$$a_1^x = a_2^{-x}$$

$$a_1^x = \left(\frac{1}{a_2}\right)^x$$

$$a_1 = \frac{1}{a_2}$$

Can we have symmetric curves? When?



$$y = \left(\frac{17}{20}\right)^x \qquad y = \left(\frac{20}{17}\right)^x$$

Immediate properties

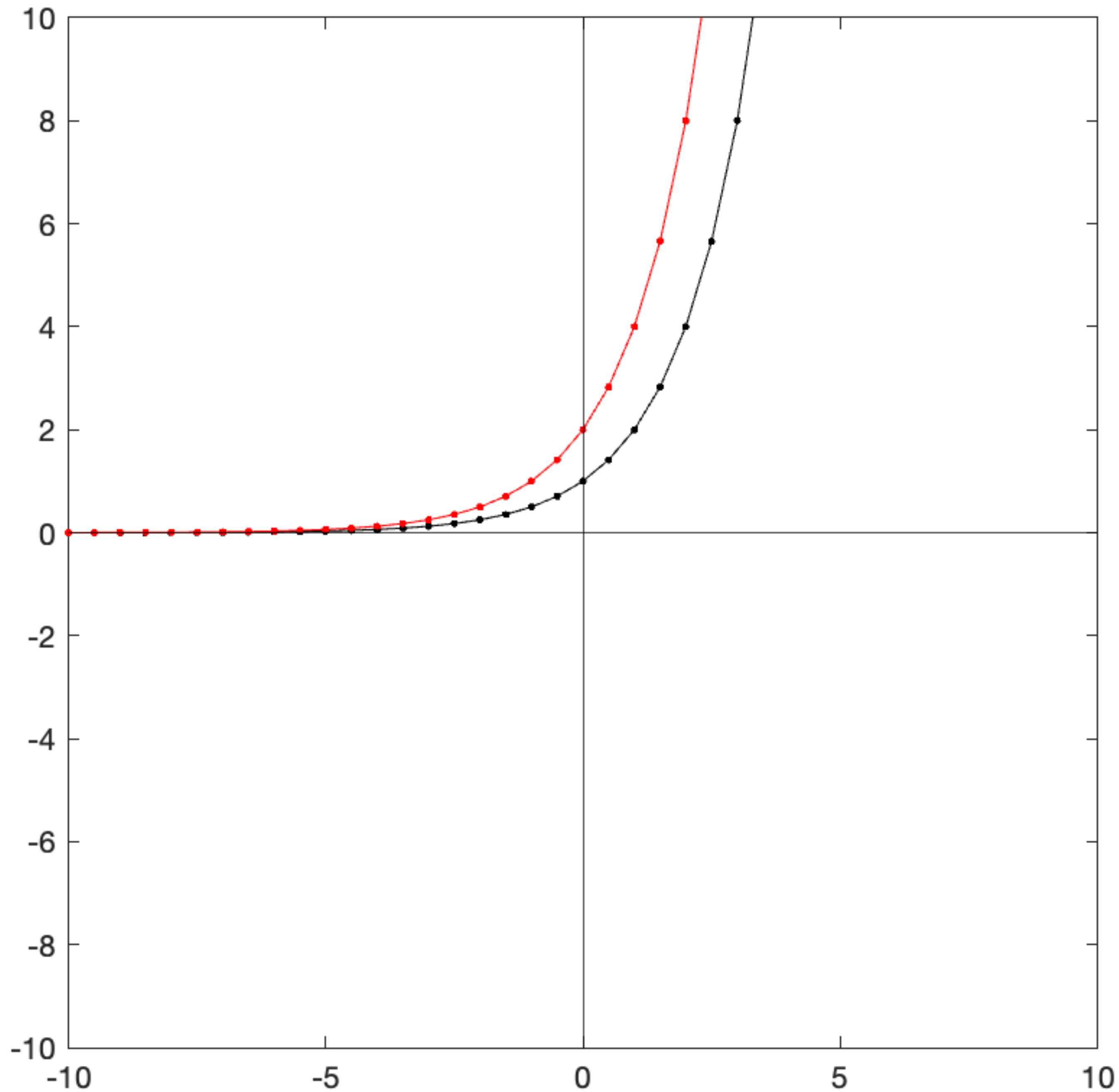
The base - a single parameter - entirely specifies the curve

Self-similarity: scaling the curve is identical to translating it

$$y(x) = 2^x$$

$$y_1(x) = 2^x \times 2 = 2^{x+1}$$

$$y_1(x) = y(x+1)$$



$y(x) = 2^x$ $\quad$ $y_1(x) = 2y(x) = y(x+1)$

Immediate properties

The base - a single parameter - entirely specifies the curve

Self-similarity: scaling the curve is identical to translating it

$$y(x) = 2^x$$

$$y_1(x) = 2^x \times 2 = 2^{x+1}$$

$$y_1(x) = y(x+1)$$

$$y(x) = a^x$$

$$y_1(x) = a^x \times b \qquad a^k = b$$

$$y_1(x) = a^x a^k = a^{x+k} = y(x+k)$$

Immediate properties

The base - a single parameter - entirely specifies the curve

Self-similarity: scaling the curve is identical to translating it

I.e., specifying a "starting size" is the same as just going to the "starting size" portion of the original curve

$$y(x) = a^x \qquad y_1(x) = Pa^x \qquad P = a^k$$

$$y_1(x) = a^k a^x$$

$$y_1(x) = a^{k+x}$$

$$y_1(x) = y(x+k)$$

Immediate properties

The base - a single parameter - entirely specifies the curve

Self-similarity: scaling the curve is identical to translating it

I.e., specifying a "starting size" is the same as just going to the "starting size" portion of the original curve

All exponential plots are tilted, the upward side is always unbounded, the downward side is always bounded

$$S = \sum_{i=0}^{n} r^i$$

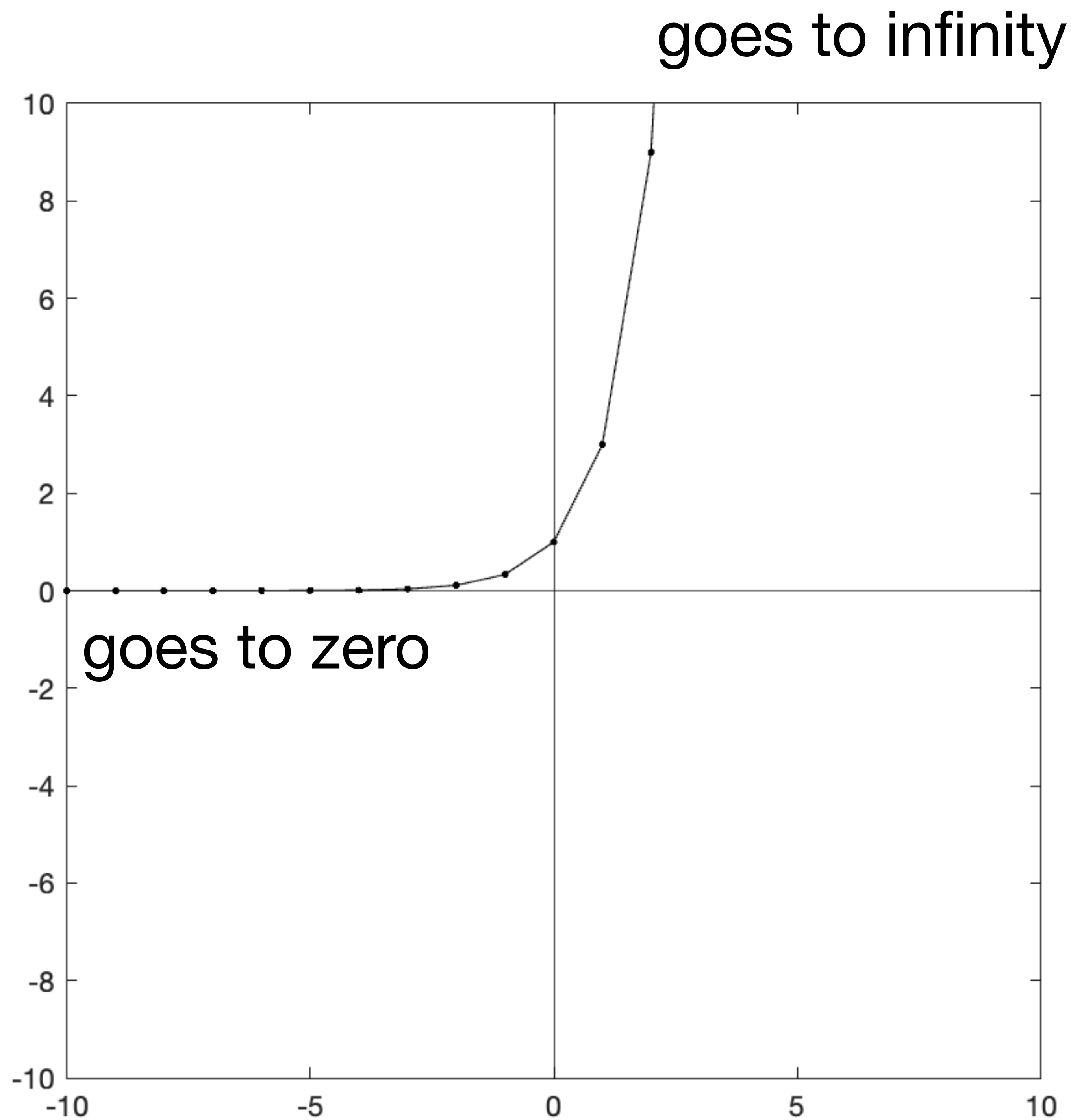
$$rS = \sum_{i=0}^{n} r^{i+1}$$

$$rS - S = r^{n+1} - 1$$

$$S = \frac{r^{n+1} - 1}{r - 1}$$

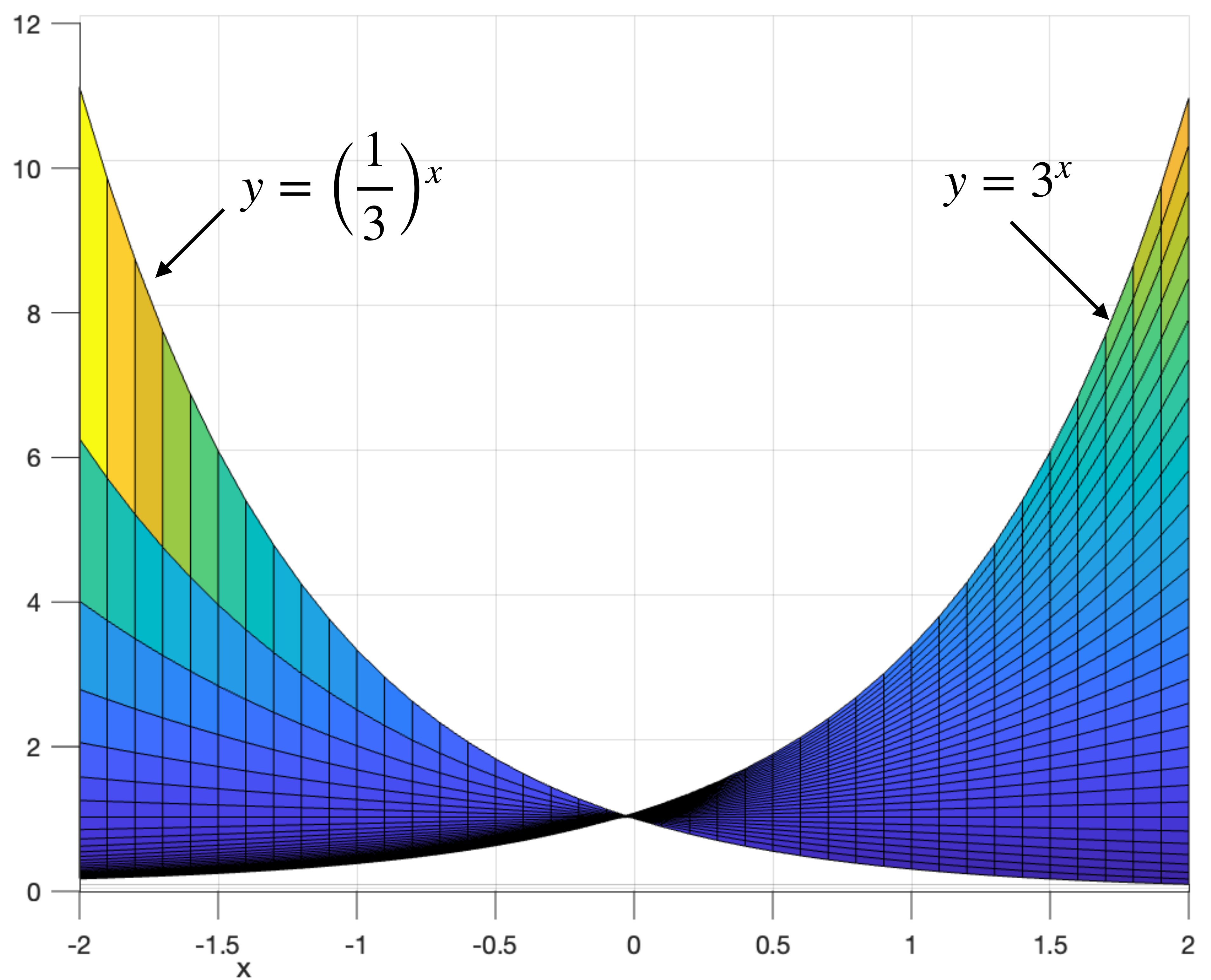
$$= \frac{1 - r^{n+1}}{1 - r}$$

$$\xrightarrow[r<1,\ n \to \text{inf}]{} \frac{1}{1 - r}$$

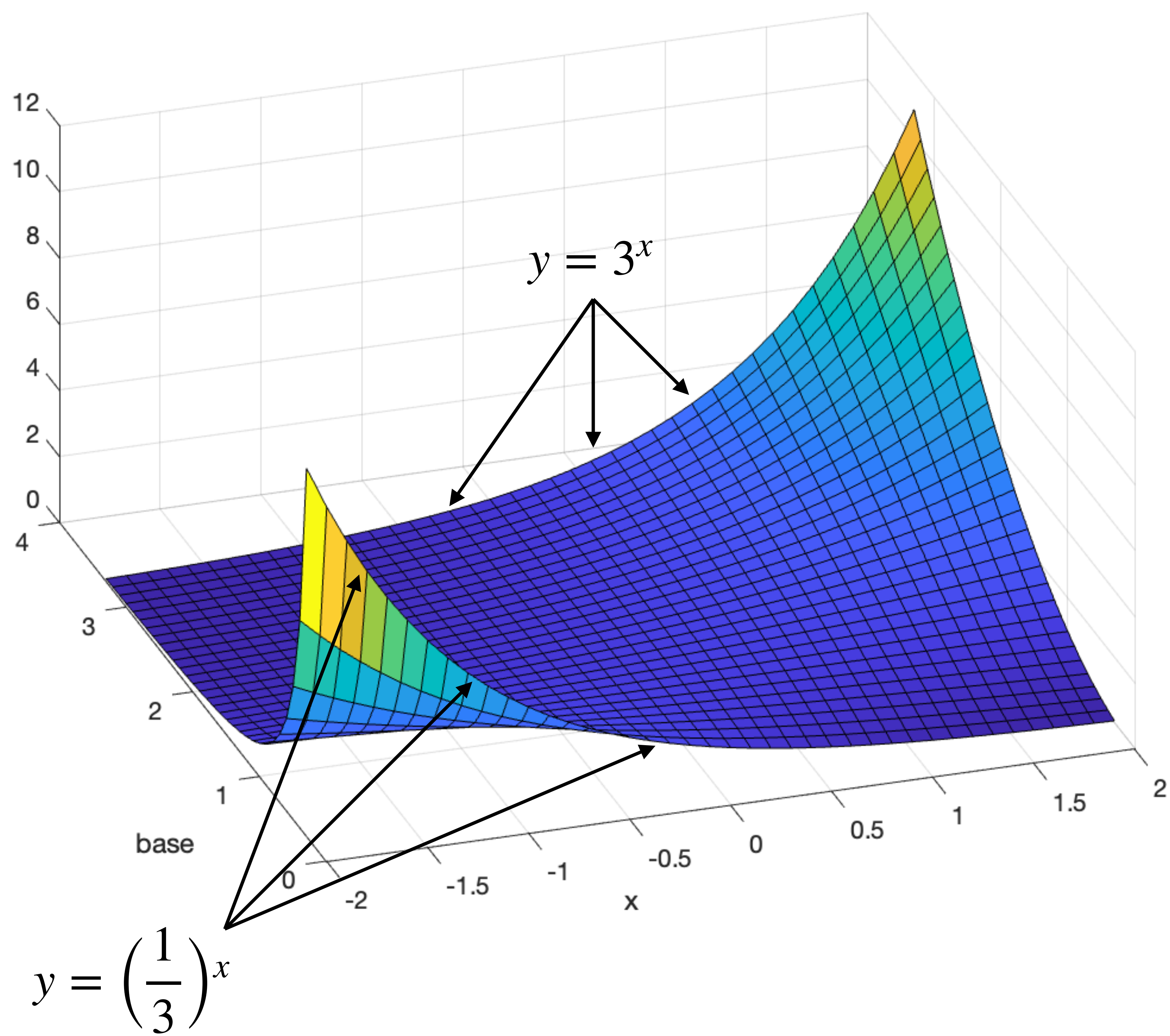


$$y = 3^x$$

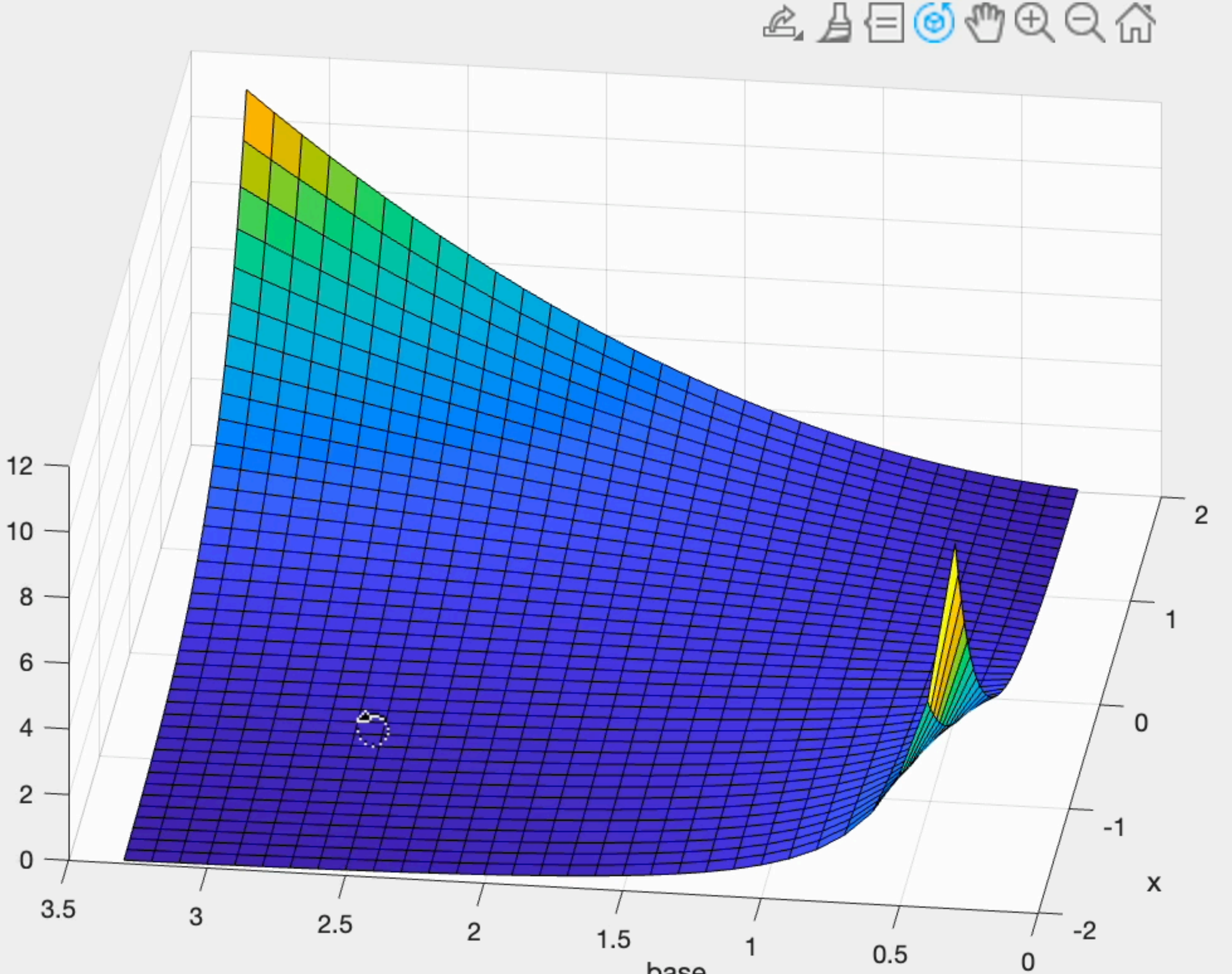
$$S = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$



12
10
8
6
4
2
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
$y = \left(\frac{1}{3}\right)^x$
$y = 3^x$



$y = 3^x$
$y = \left(\frac{1}{3}\right)^x$
base
x
12
10
8
6
4
2
0
4
3
2
1
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2


12
10
8
6
4
2
0
3.5
3
2.5
2
1.5
1
0.5
0
base
2
1
0
-1
-2
x

Immediate properties

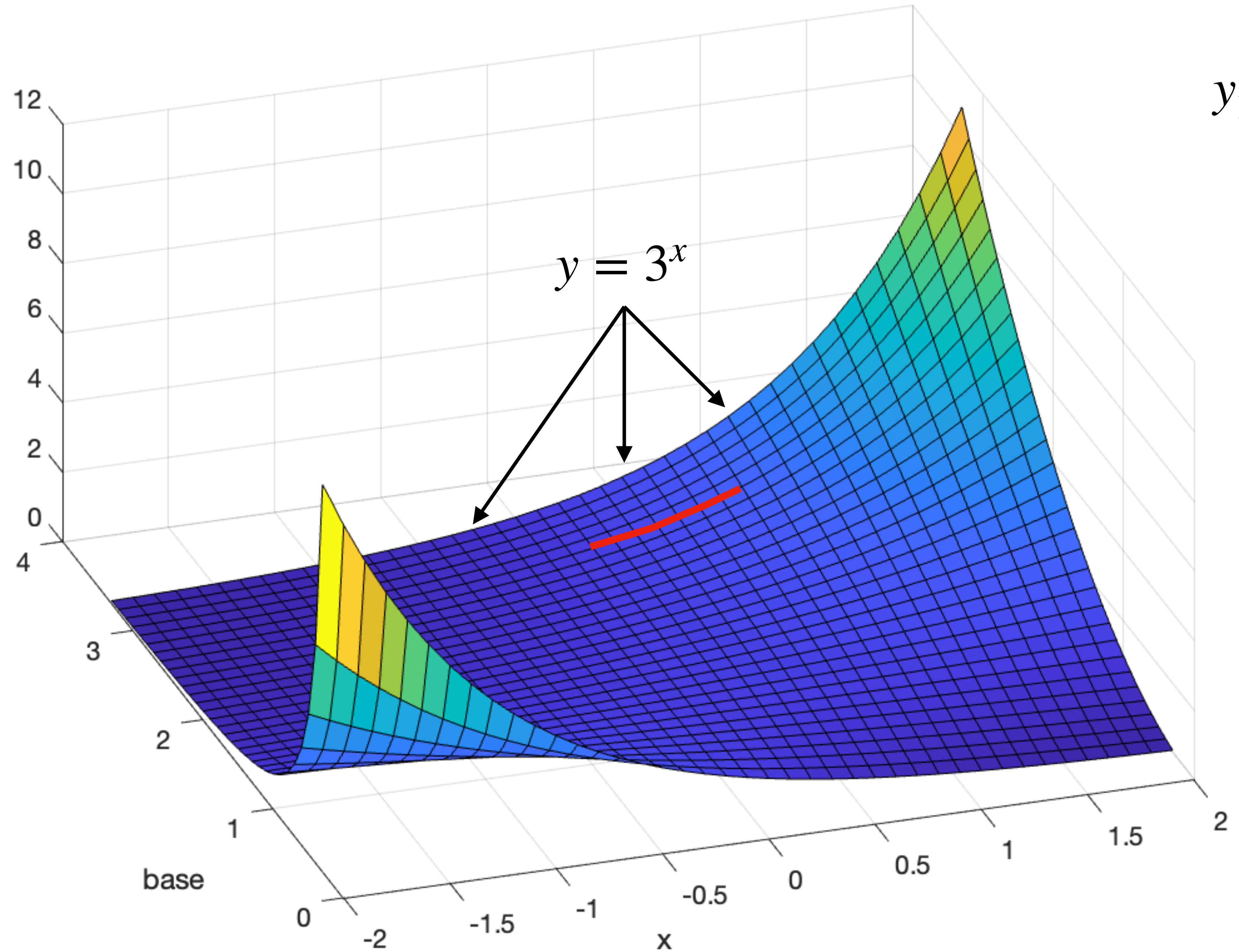
The base - a single parameter - entirely specifies the curve

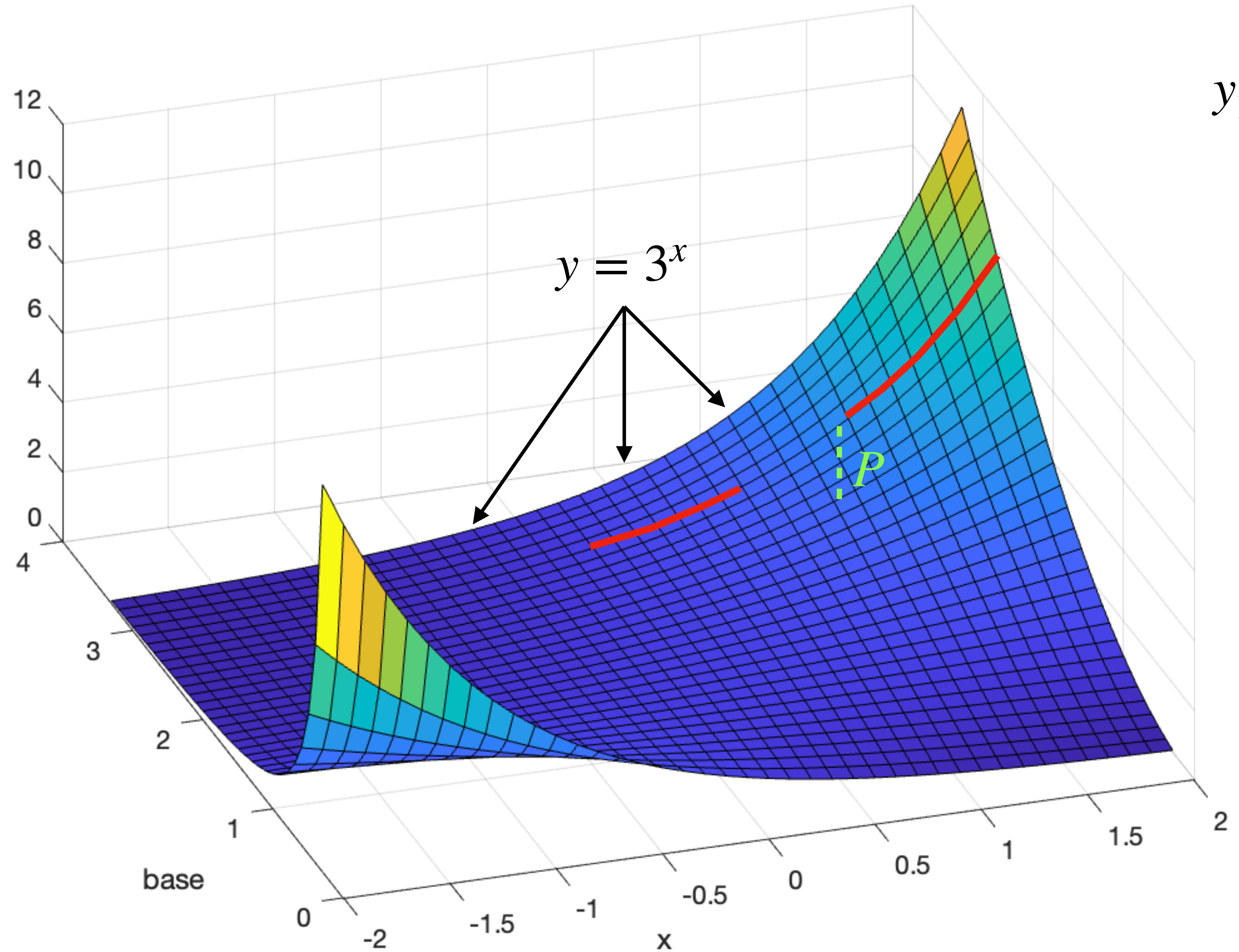
Self-similarity: scaling the curve is identical to translating it

I.e., specifying a "starting size" is the same as just going to the "starting size" portion of the original curve

All exponential plots are tilted, the upward side is always unbounded, the downward side is always bounded

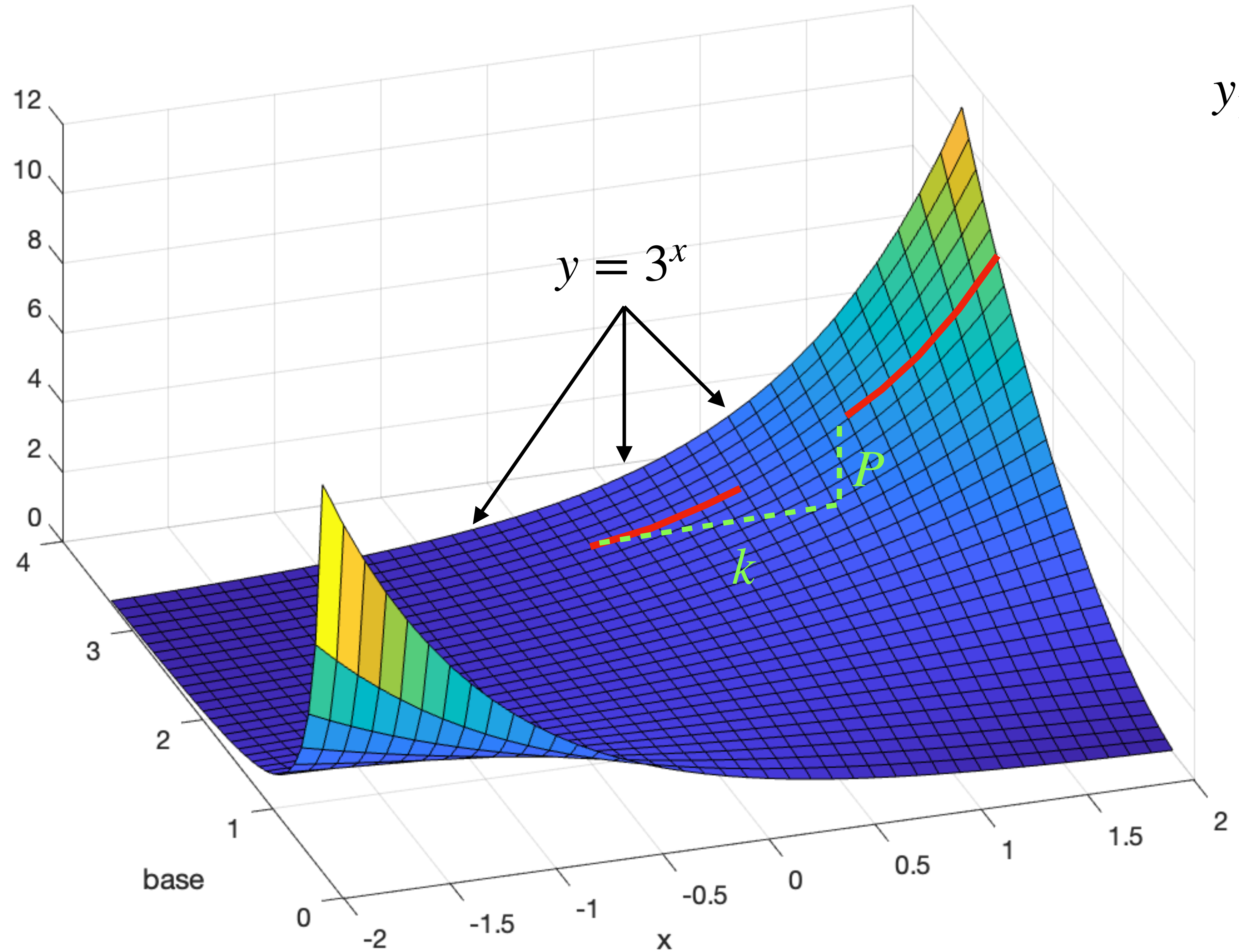
$$y_1(x) = Pa^x$$





$$y_1(x) = Pa^x$$

$$P = a^k$$



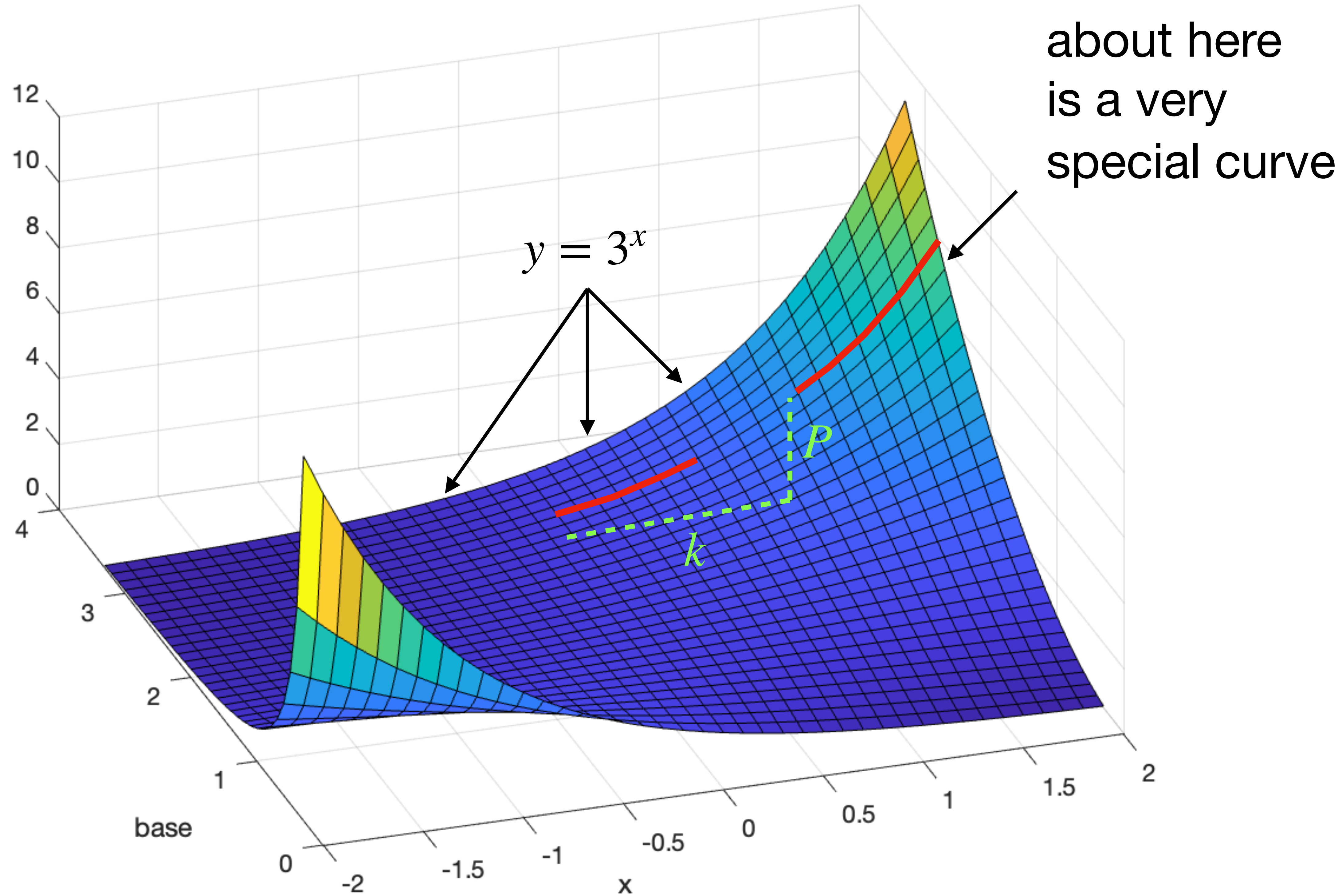
$$y_1(x) = Pa^x$$

$$P = a^k$$

Thoughts for pause

This surface is continuous over all positive a and all x

Buried in this surface is a special curve, between 2 and 3

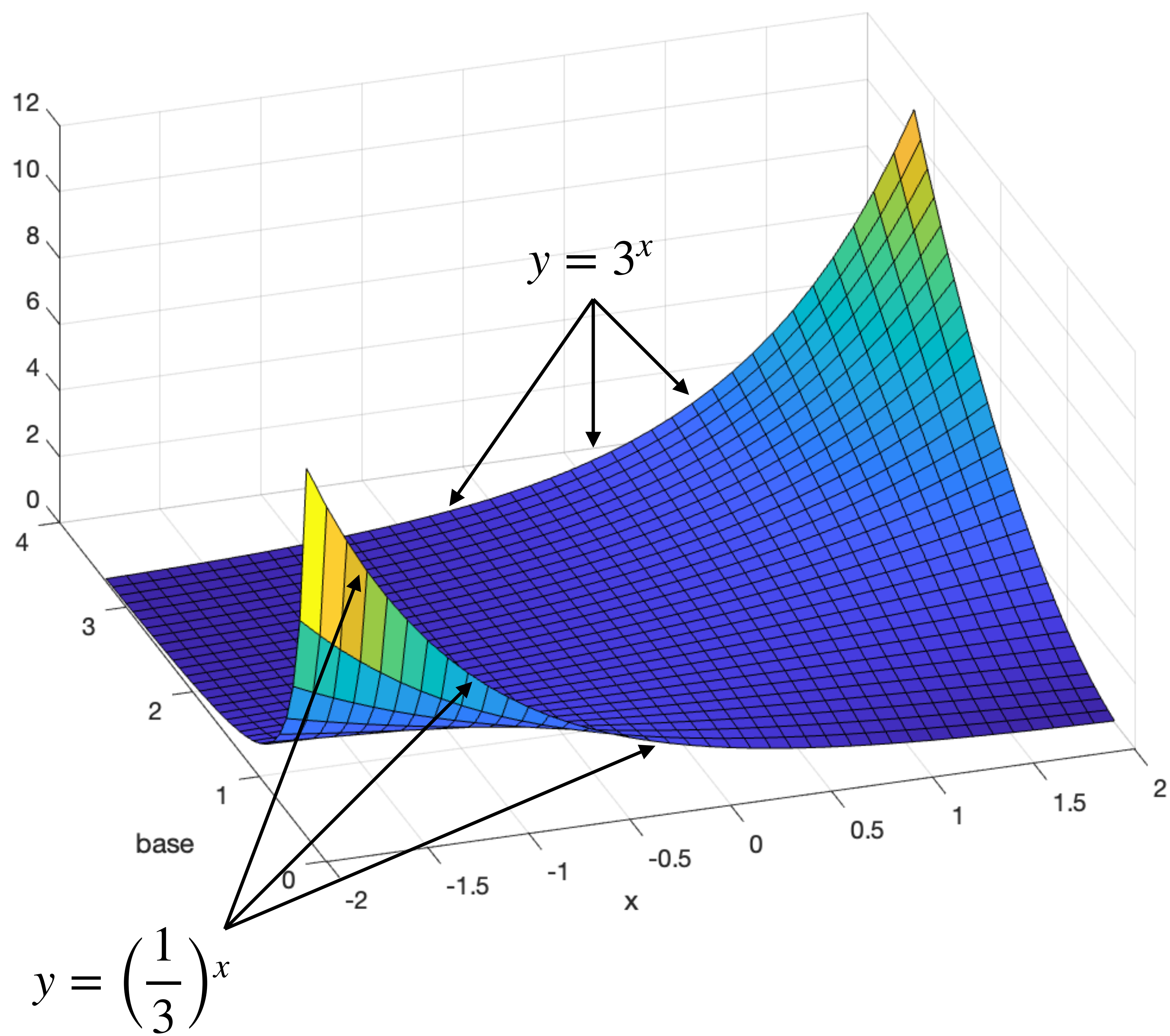

about here
is a very
special curve
$y = 3^x$
P
k
12
10
8
6
4
2
0
4
3
2
1
0
base
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x

Thoughts for pause

This surface is continuous over all positive a and all x

Buried in this surface is a special curve, between 2 and 3

One view: the surfaces are the curves of
exponential functions, a^x



$y = 3^x$
$y = \left(\frac{1}{3}\right)^x$
12
10
8
6
4
2
0
4
3
2
1
0
base
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x

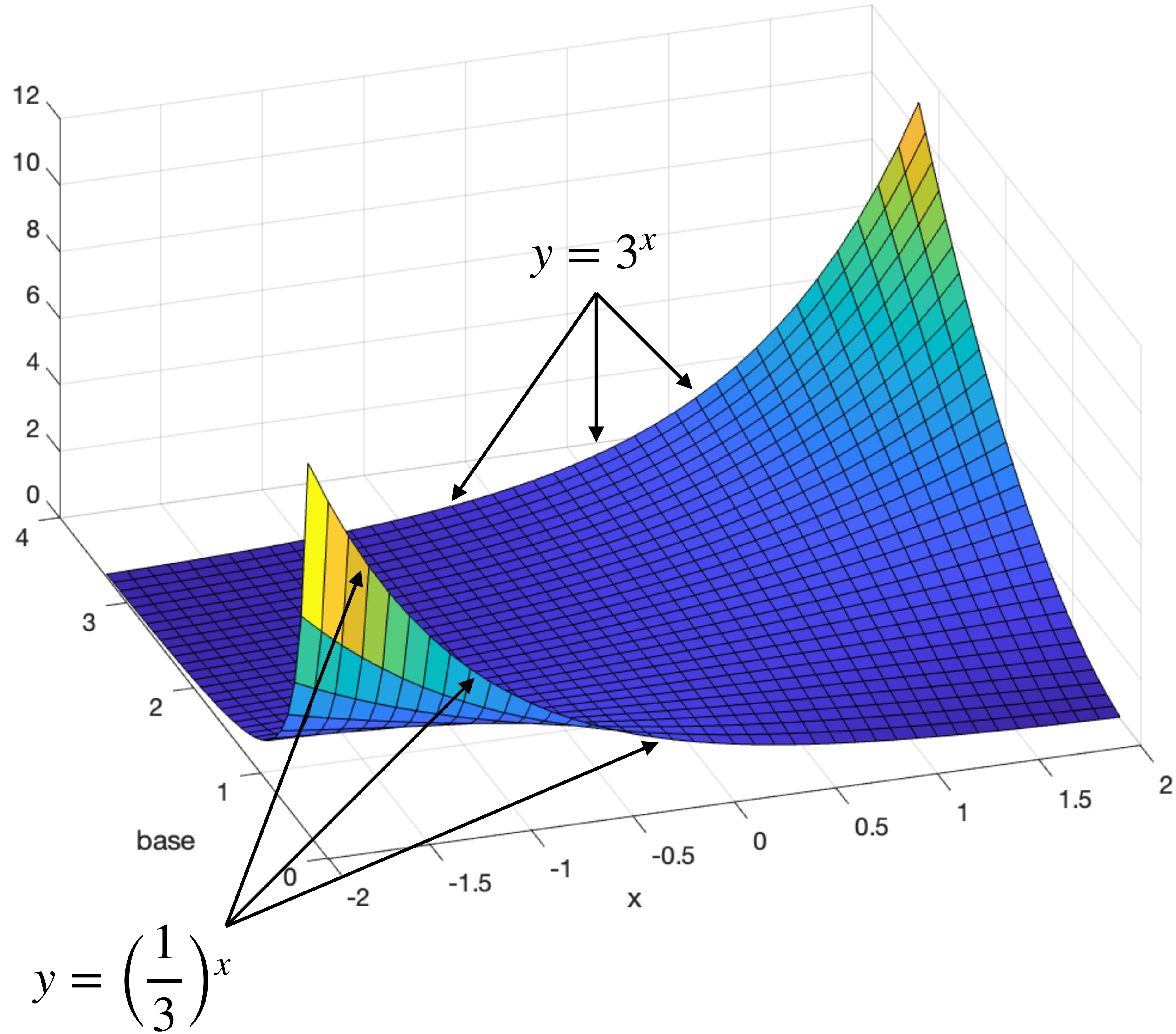
Thoughts for pause

This surface is continuous over all positive a and all x

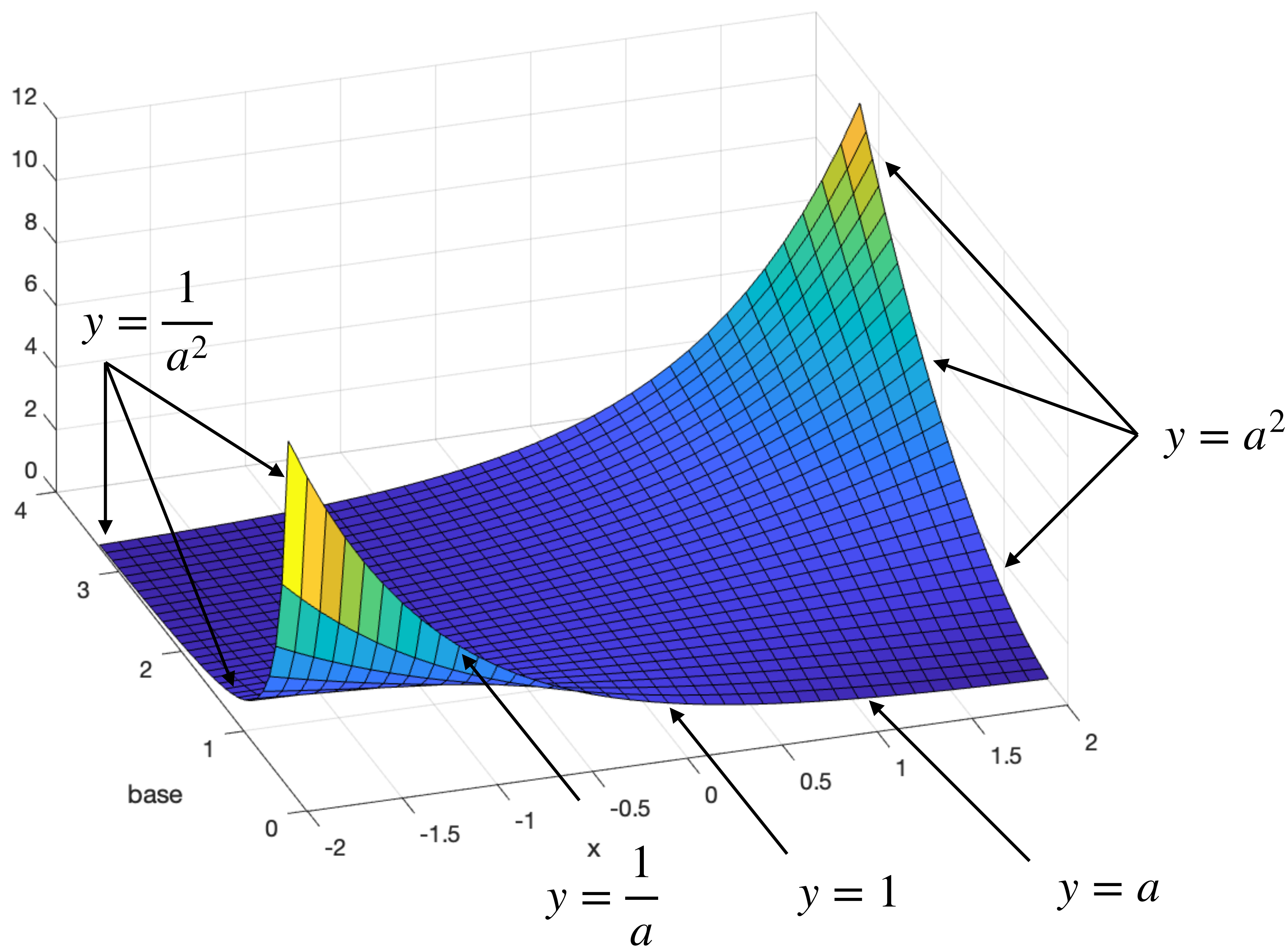
Buried in this surface is a special curve, between 2 and 3

One view: the surfaces are the curves of exponential functions, a^x

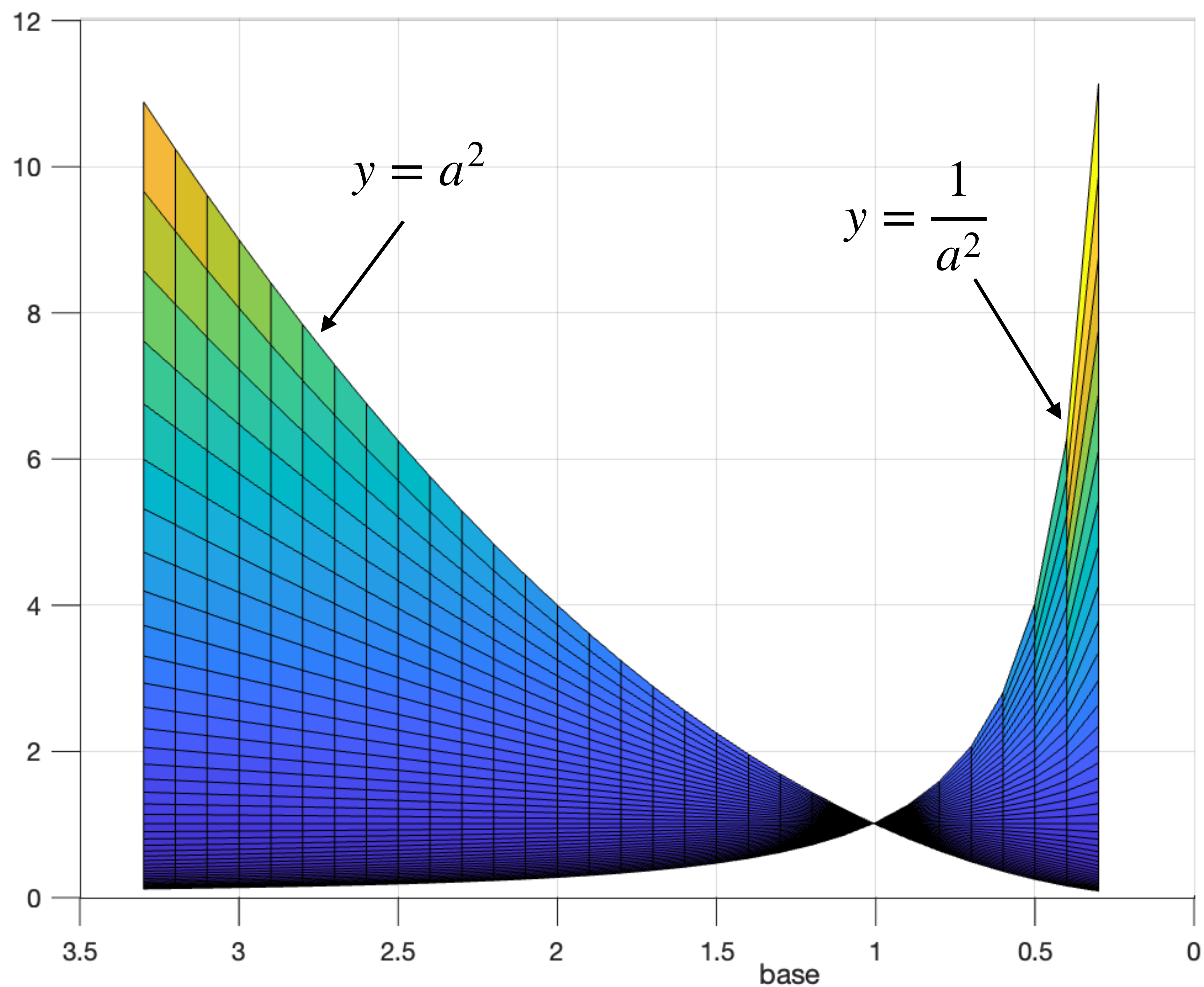
Orthogonal view: the surfaces are curves of "continuous" polynomials x^a



$y = 3^x$
$y = \left(\frac{1}{3}\right)^x$
12
10
8
6
4
2
0
4
3
2
1
0
base
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x



12
10
8
6
4
2
0
4
3
2
1
0
base
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
$y = \frac{1}{a^2}$
$y = a^2$
$y = \frac{1}{a}$
$y = 1$
$y = a$


12
10
8
6
4
2
0
3.5
3
2.5
2
1.5
1
0.5
0
base
$y = a^2$
$y = \frac{1}{a^2}$

Thoughts for pause

This surface is continuous over all positive a and all x

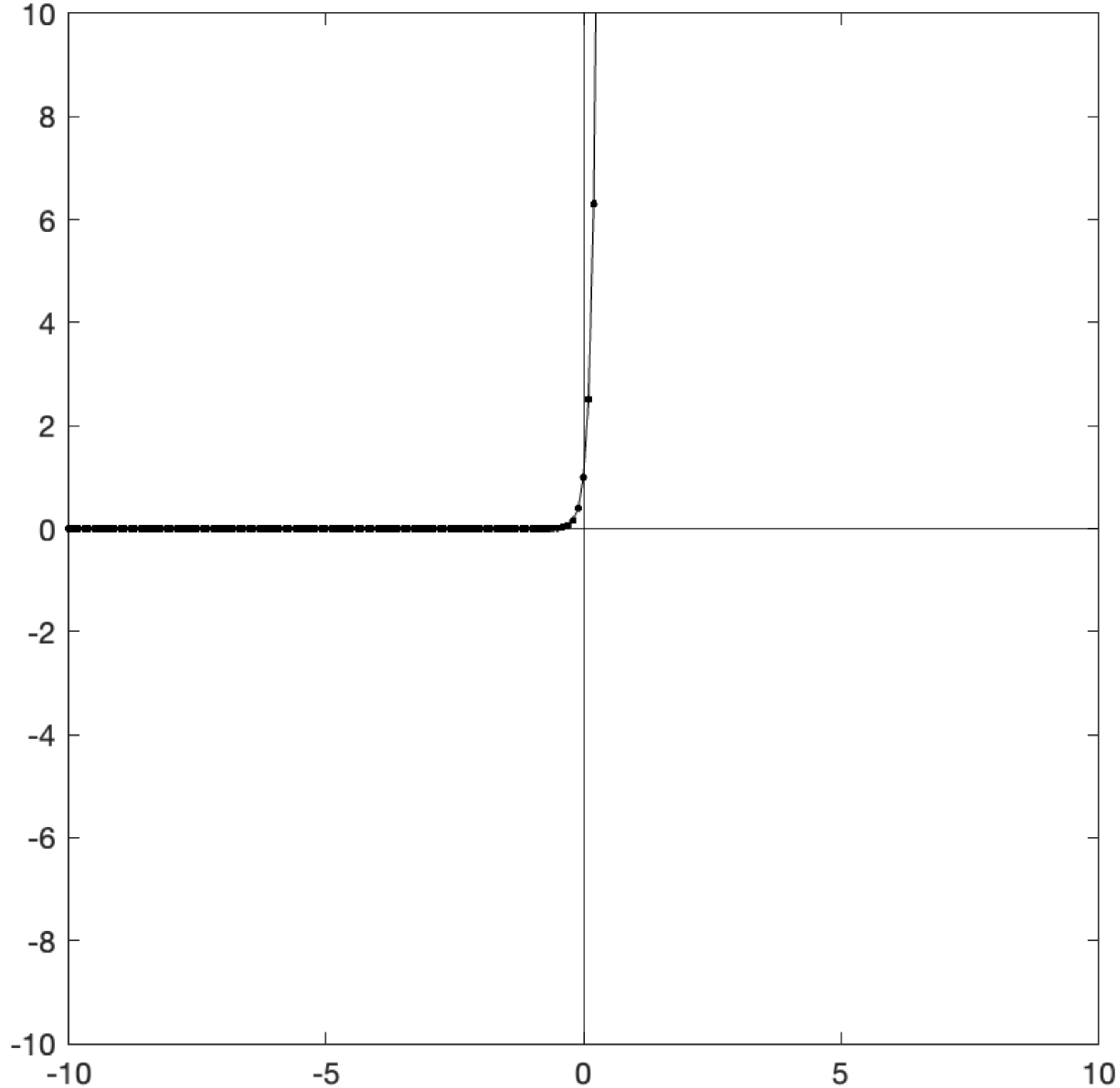
Buried in this surface is a special curve, between 2 and 3

One view: the surfaces are the curves of
exponential functions, a^x

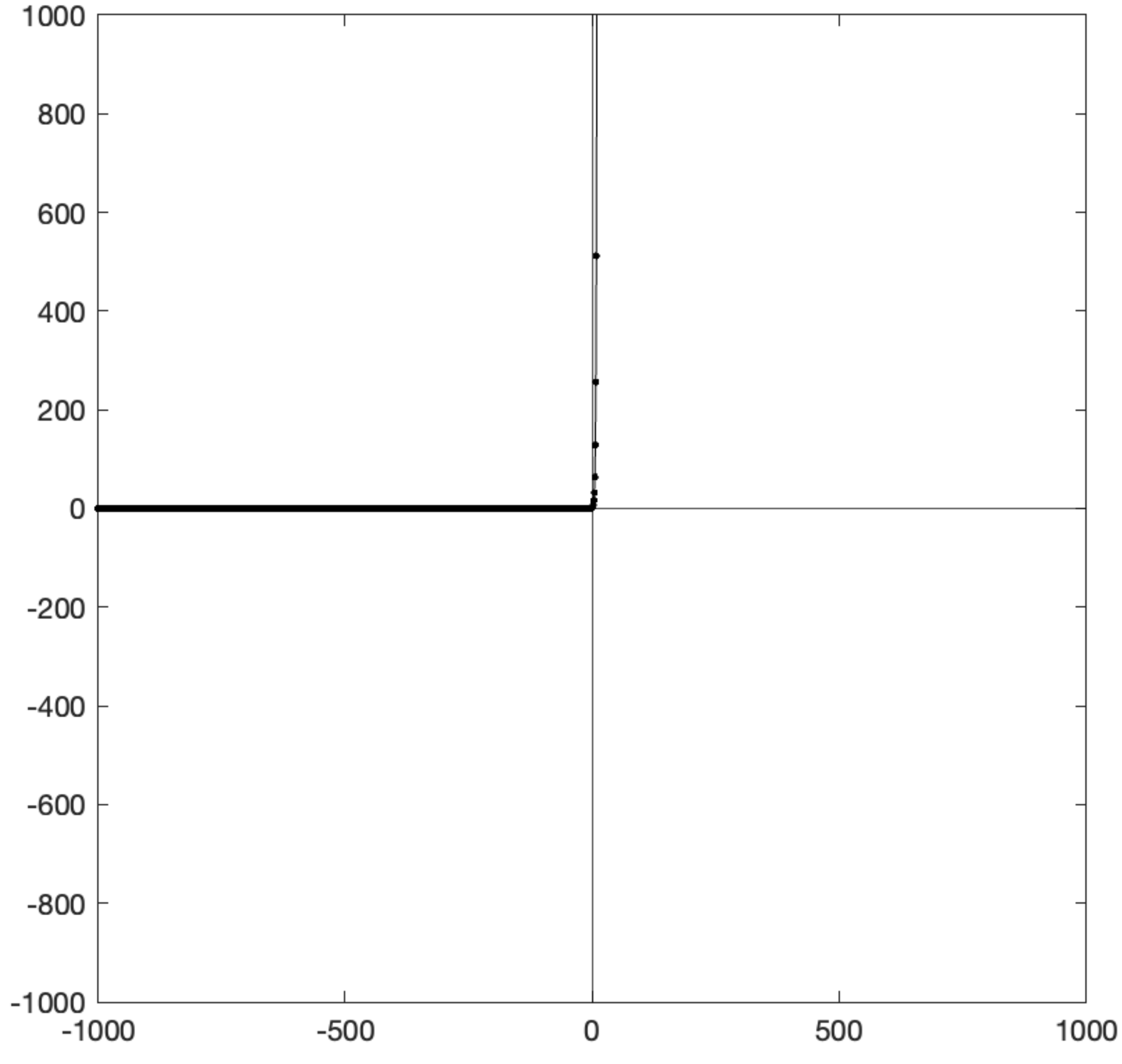
Orthogonal view: the surfaces are curves of
"continuous" polynomials x^a

Exponential functions grow fast

$y = 1000^x$



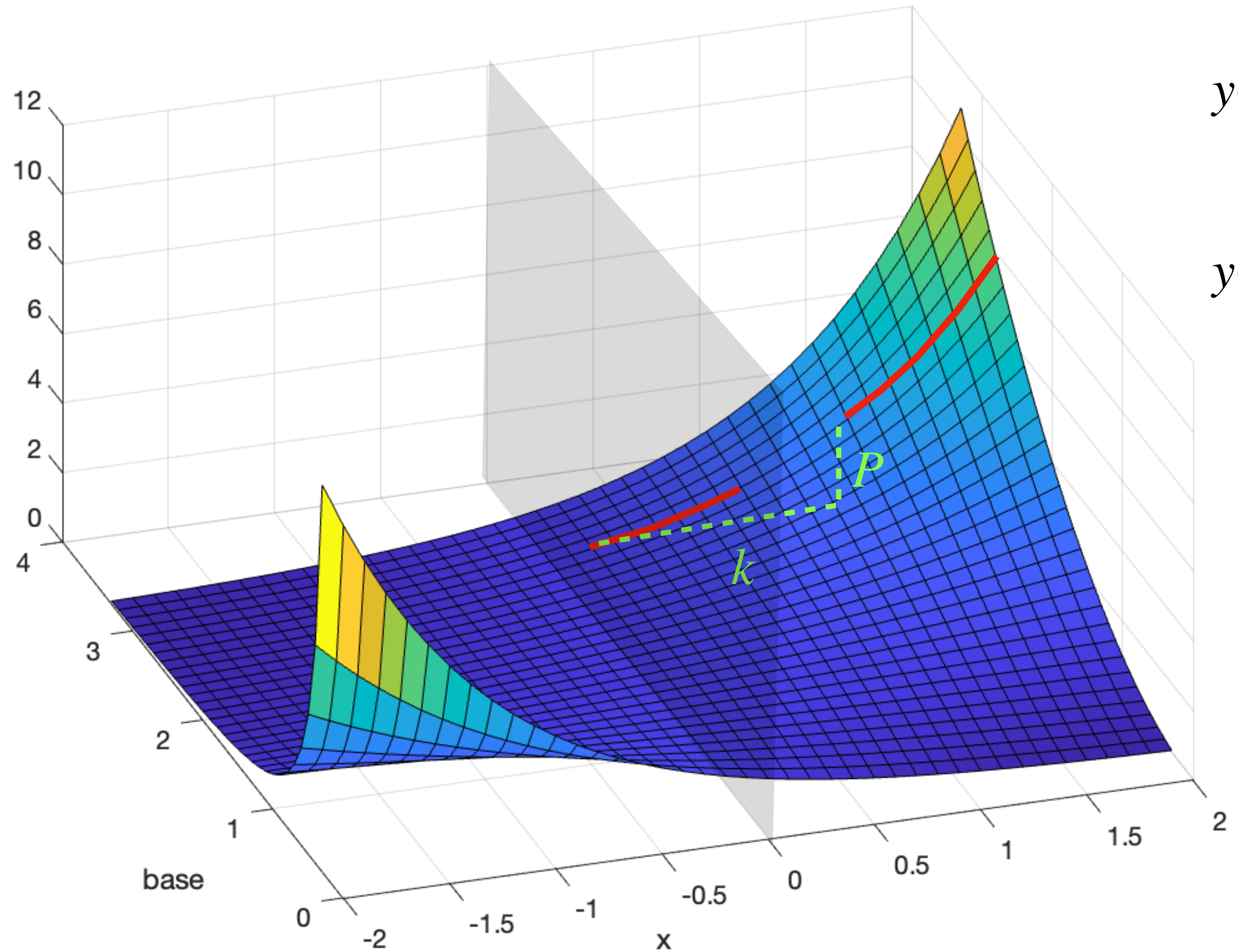
$$y = 2^x$$



END PART 1

PART 2

We saw that a single exponential curve is self-similar

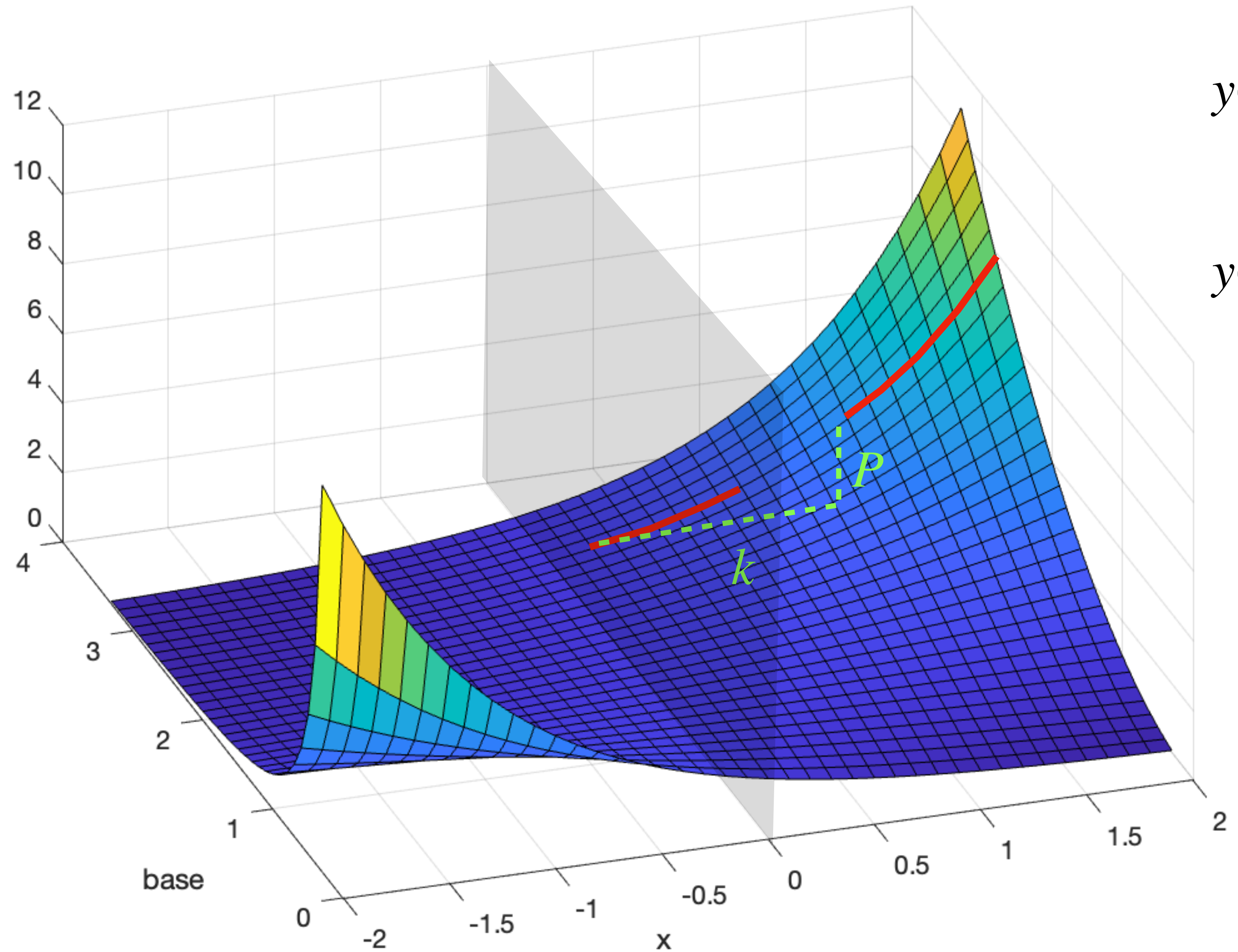


$$y(x) = Pa^x$$

$$P = a^k$$

$$y(x) = a^{k+x}$$

Said differently: each part is a scaled version of another part

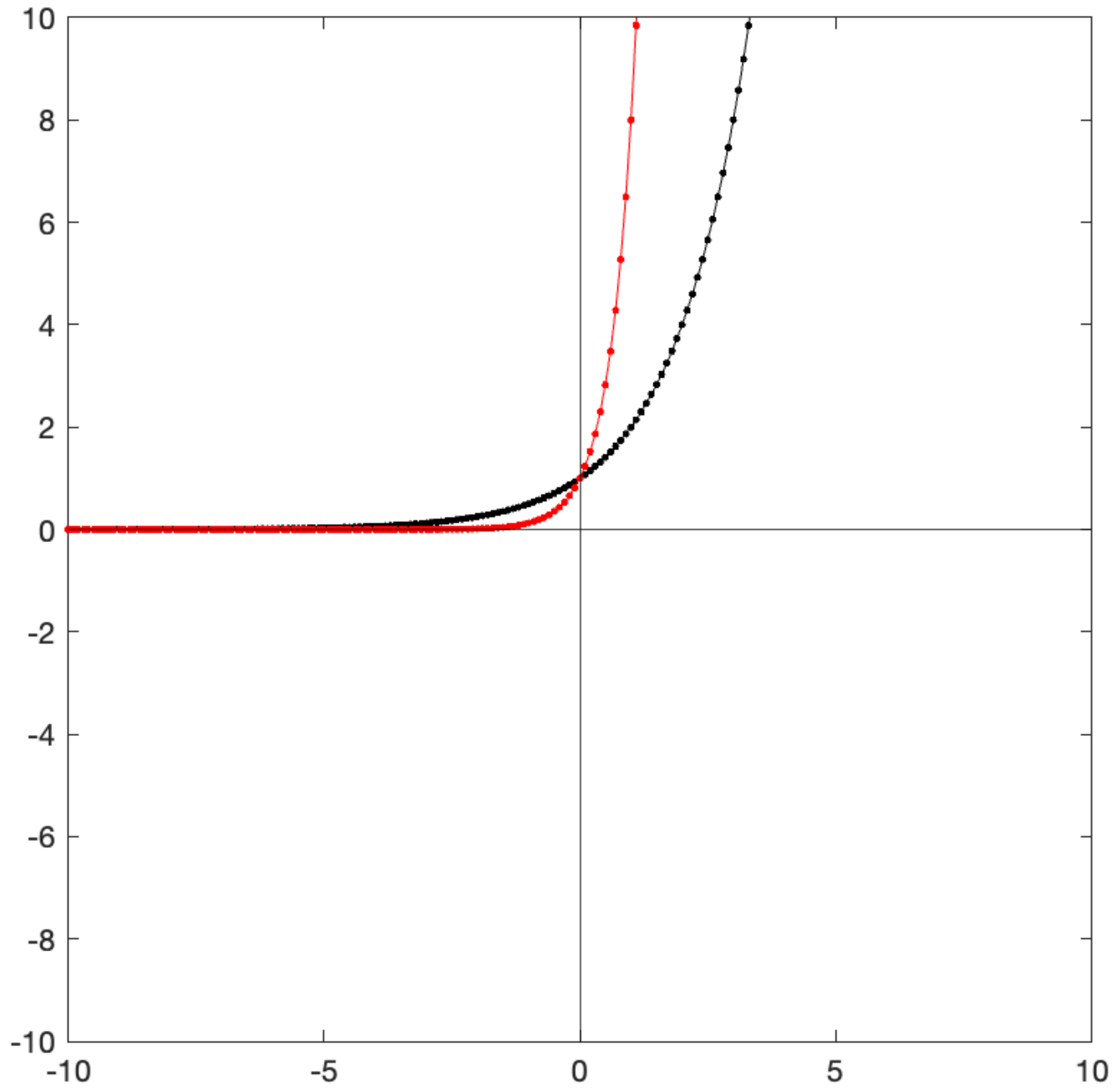


$$y(x) = Pa^x$$

$$P = a^k$$

$$y(x) = a^{k+x}$$

How do different exponential curves relate?



$y = 2^x$ $\quad$ $y = 8^x$

All exponential curves are fundamentally a single curve

$$y_1 = a_1^x \qquad y_2 = a_2^x$$

$$a_1 = a_2^k$$

$$a_1^x = (a_2^k)^x$$

$$a_1^x = a_2^{kx}$$

Any exponential curve can be expressed as another

All we must do is scale the x-axis appropriately - by k

All exponential curves are fundamentally a single curve

$$y_1 = 2^x \qquad y_2 = 8^x$$

$$2 = 8^{\frac{1}{3}} \qquad 8 = 2^3$$

$$2^x = (8^{\frac{1}{3}})^x \qquad 8^x = (2^3)^x$$

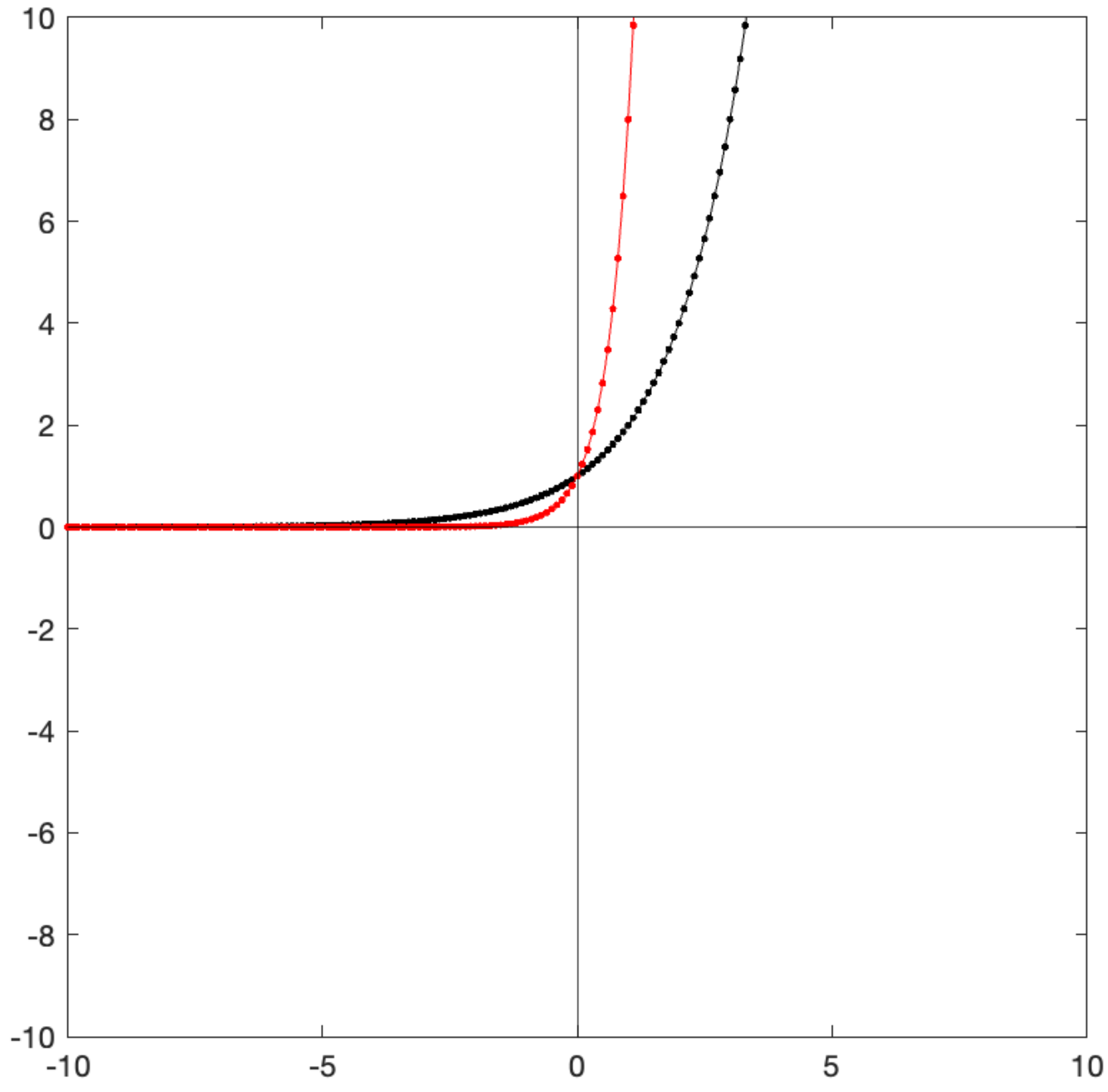
$$2^x = 8^{\frac{1}{3}x} \qquad 8^x = 2^{3x}$$

$$k = \frac{1}{3} \qquad k = 3$$

Any exponential curve can be expressed as another

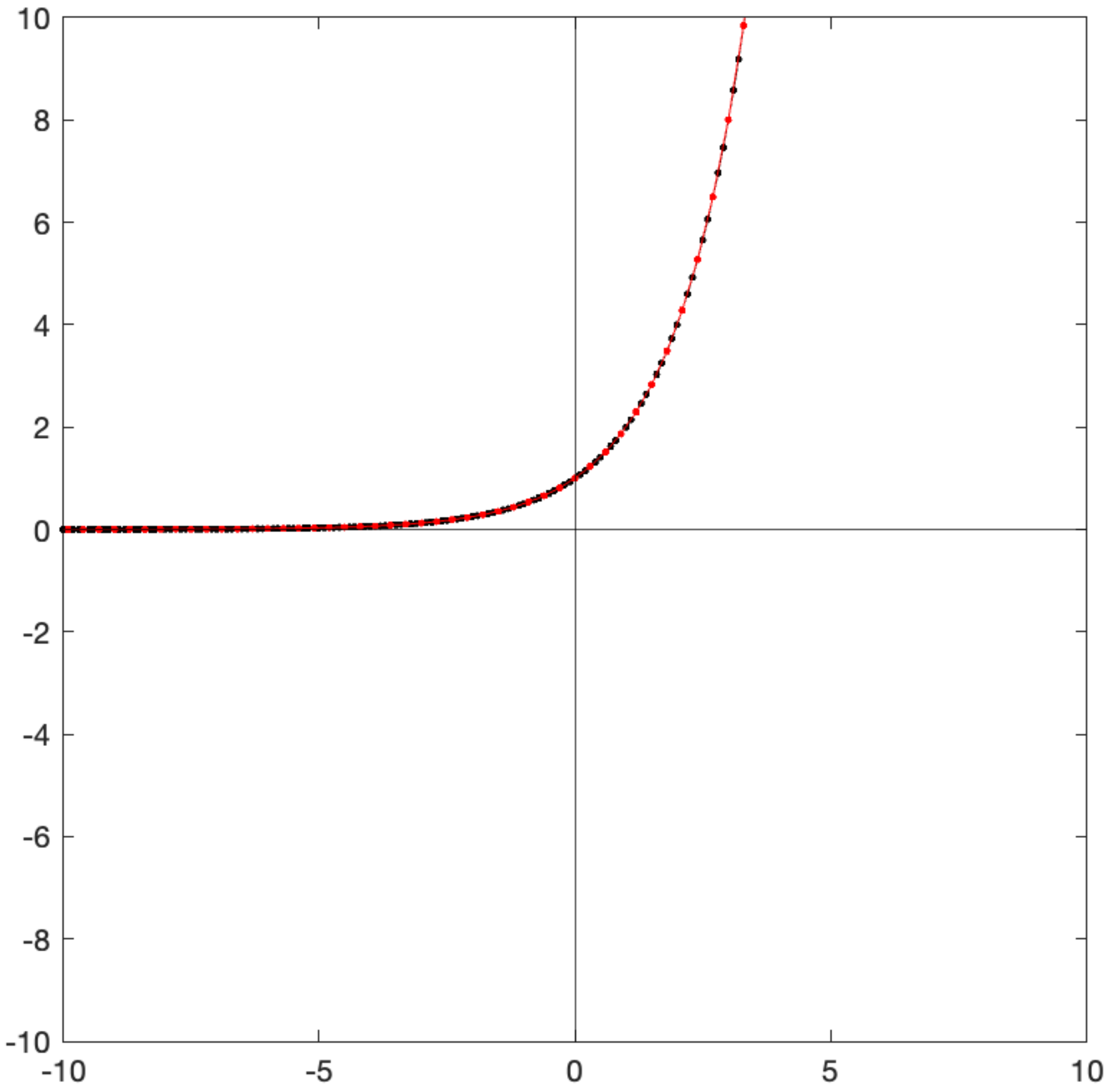
All we must do is scale the x-axis appropriately - by k

All exponential curves are fundamentally a single curve



$y = 2^x$ $\color{red}{y = 8^x}$

All exponential curves are fundamentally a single curve

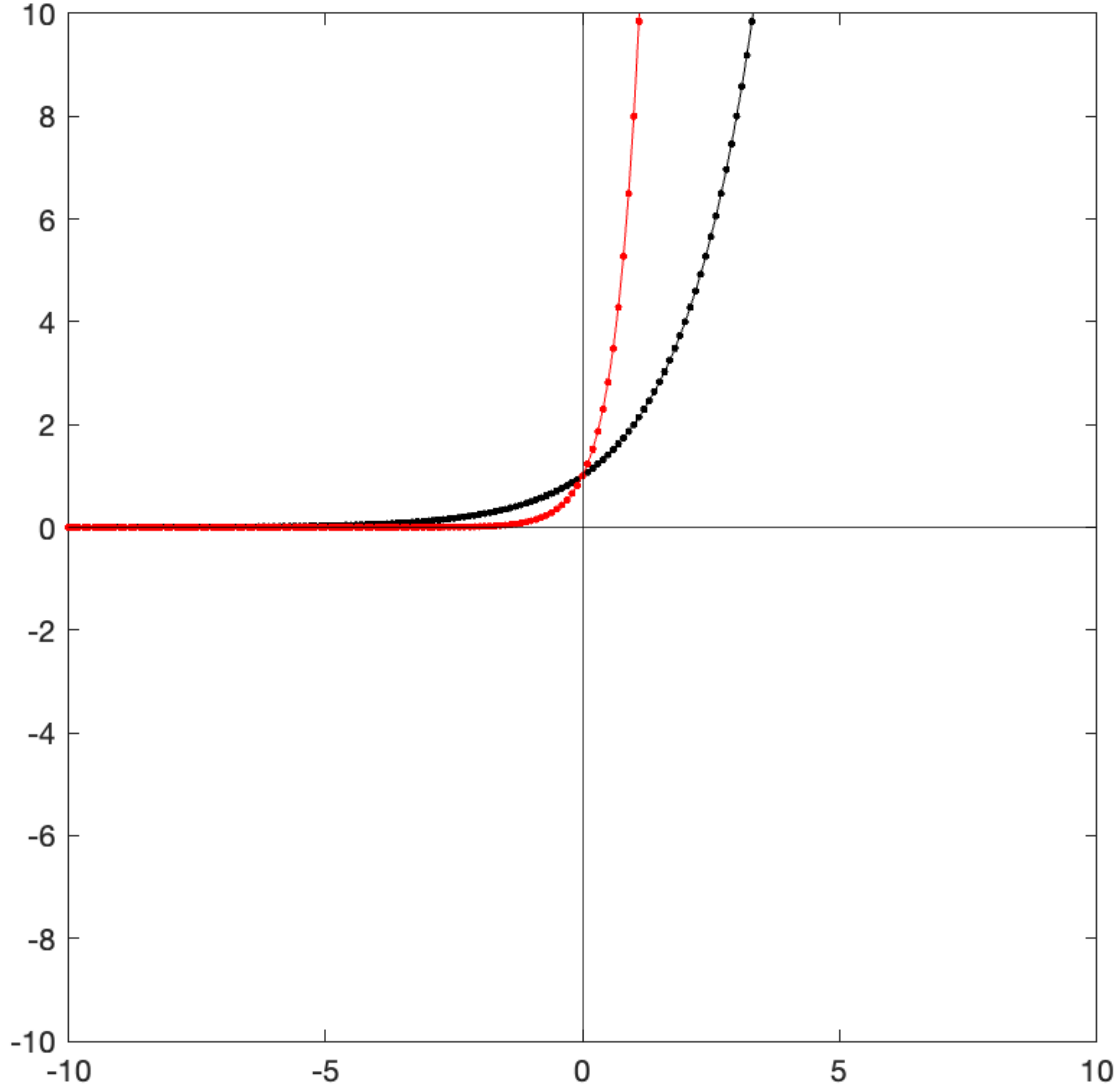


$y = 2^x$

$y = 8^x$

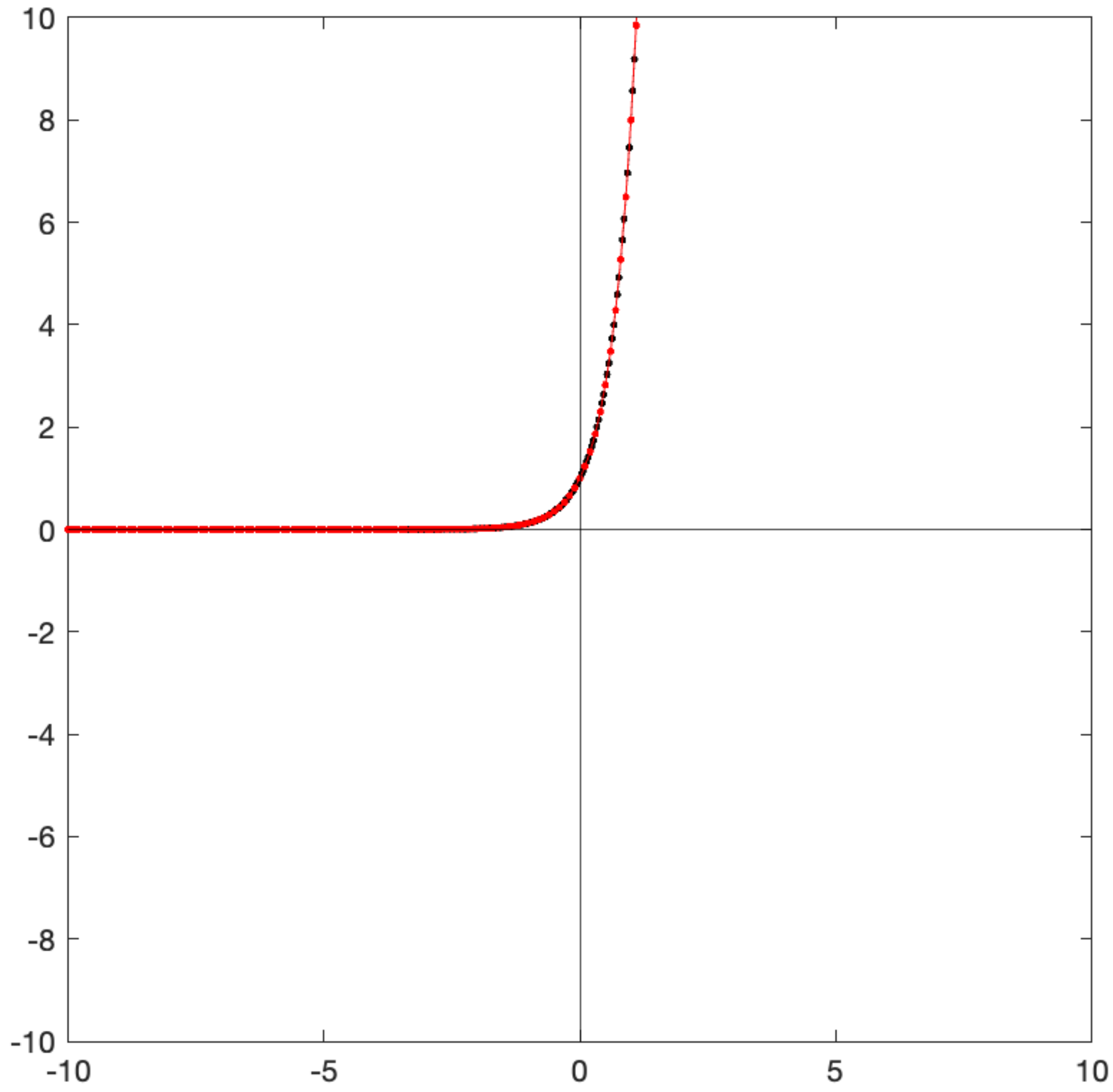
Plotting on 3x

All exponential curves are fundamentally a single curve



$y = 2^x$ $\quad$ $y = 8^x$

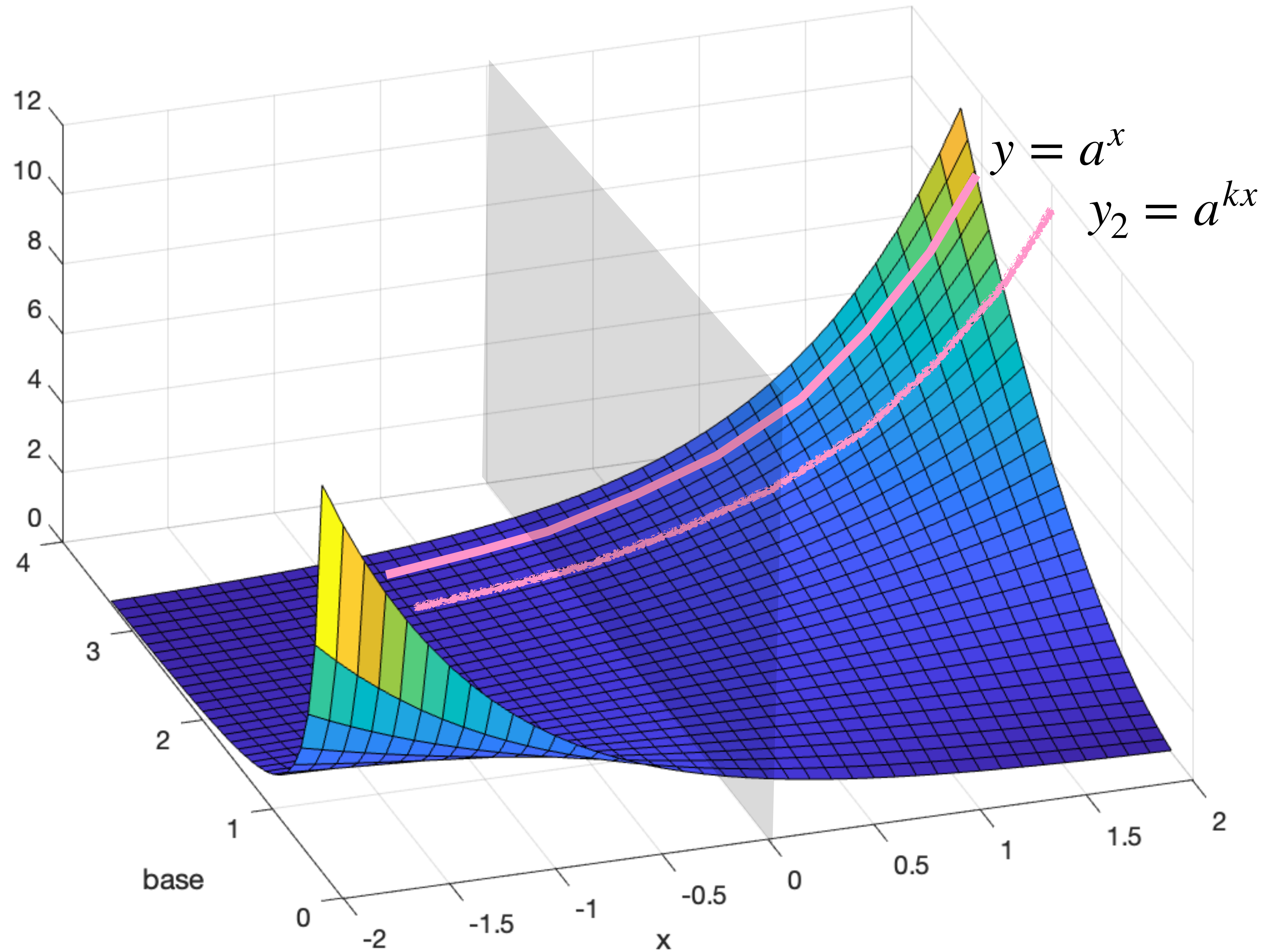
All exponential curves are fundamentally a single curve



$y = 2^x$ $\color{red}{y = 8^x}$

Plotting on x/3

All exponential curves are fundamentally a single curve



All exponential curves are fundamentally a single curve

And sections of a curve are self-similar

All sections of all curves map to a single section of a single curve

$$y = a^x \qquad\qquad y_2 = P a_2^x$$

$$a^k = a_2$$

$$a^{k_P} = P$$

$$y_2 = a^{kx + k_P}$$

Suppose we chose a single base as the master base

$$y = a^{bx+c}$$

THIS COVERS ALL POSSIBLE EXPONENTIAL FUNCTIONS

All exponential curves are fundamentally a single curve

And sections of a curve are self-similar

All sections of all curves map to a single section of a single curve

$$y = a^{bx+c}$$

$a : master\ base \quad (some\ number)$

$b : conversion\ to\ other\ base \quad (stretch\ on\ x)$

$c : shift\ to\ starting\ height \quad (scale\ on\ y)$

Suppose we chose a single base as the master base

THIS COVERS ALL POSSIBLE EXPONENTIAL FUNCTIONS

All exponential curves are fundamentally a single curve

$$f(x) = 5 \times 1.4^{x}$$

$$y = a^{bx+c}$$

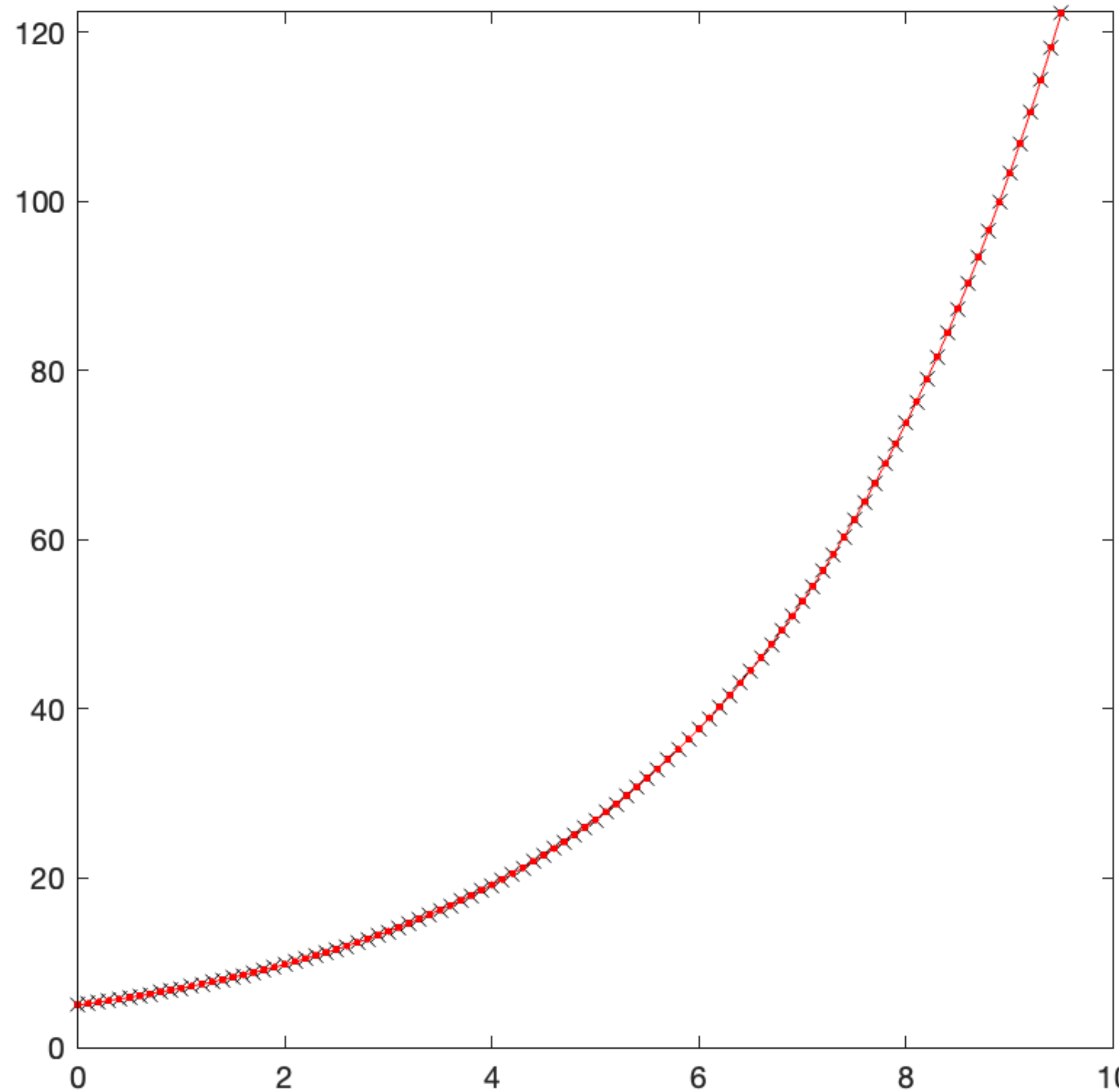
$$a = 6$$

$$a^{b} = 1.4,\ b = .19$$

$$a^{c} = 5,\ c = .89$$

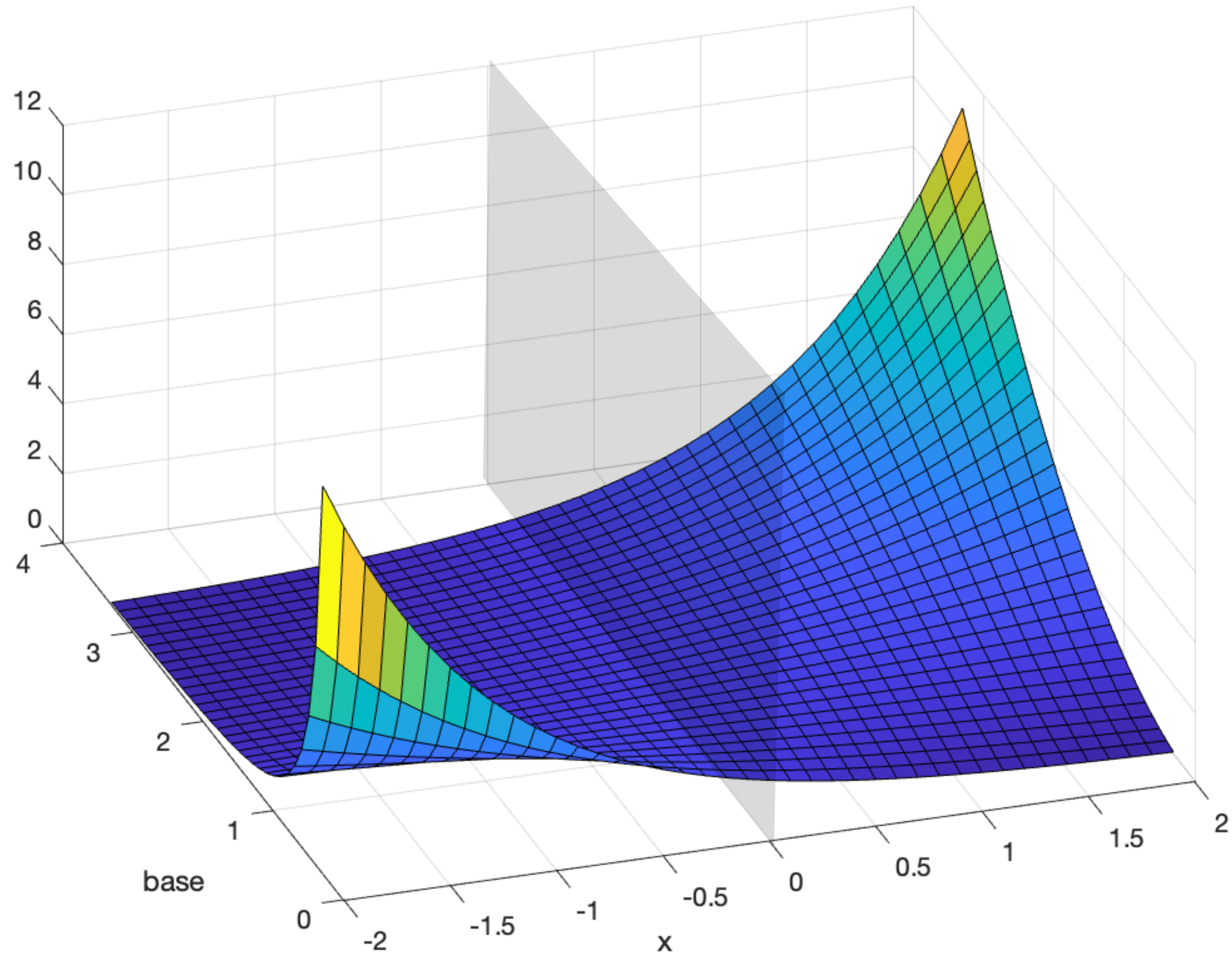
$$y = a^{bx+c}$$

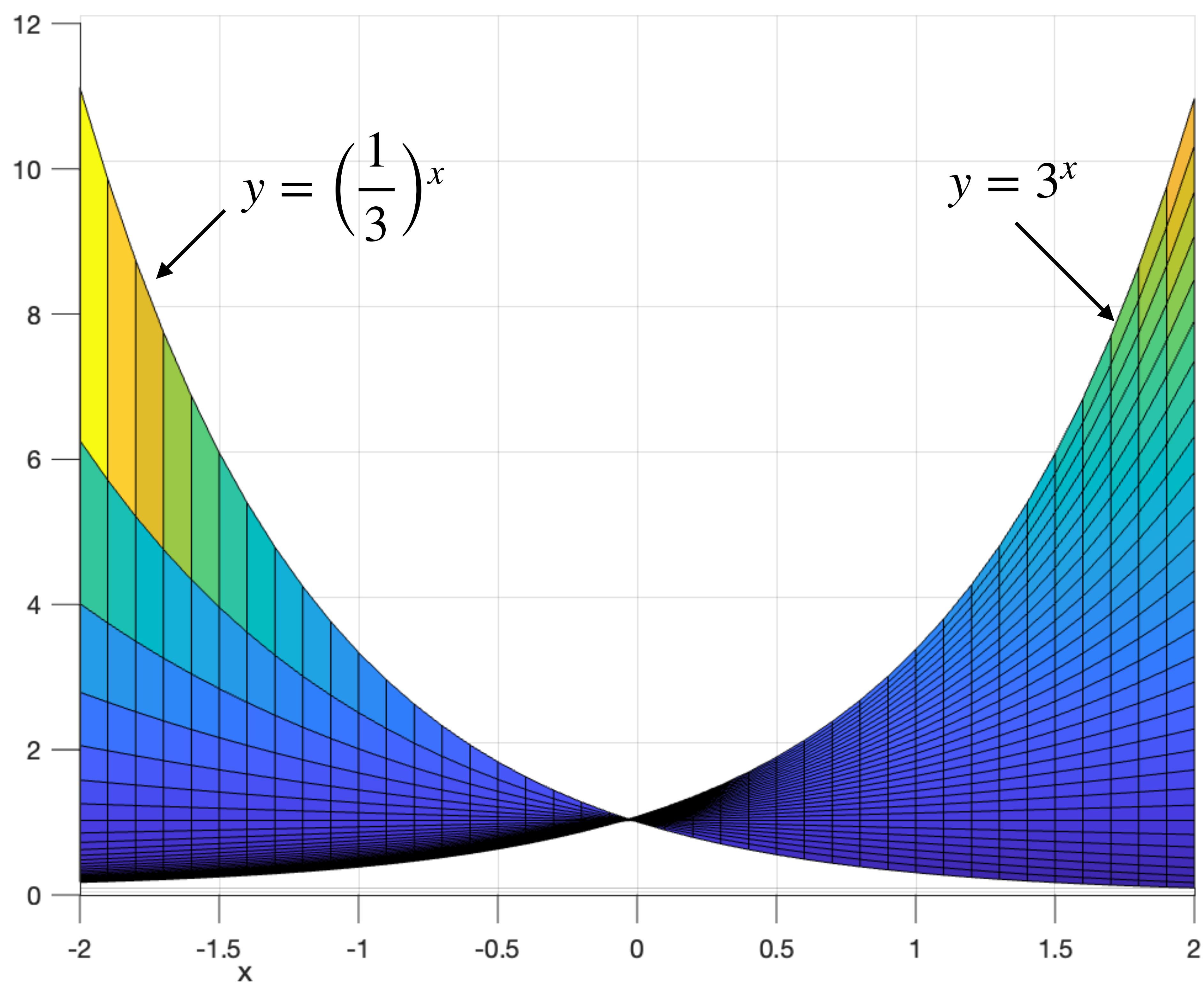
$$y = 6^{.19x+.89}$$



THIS COVERS ALL POSSIBLE EXPONENTIAL FUNCTIONS

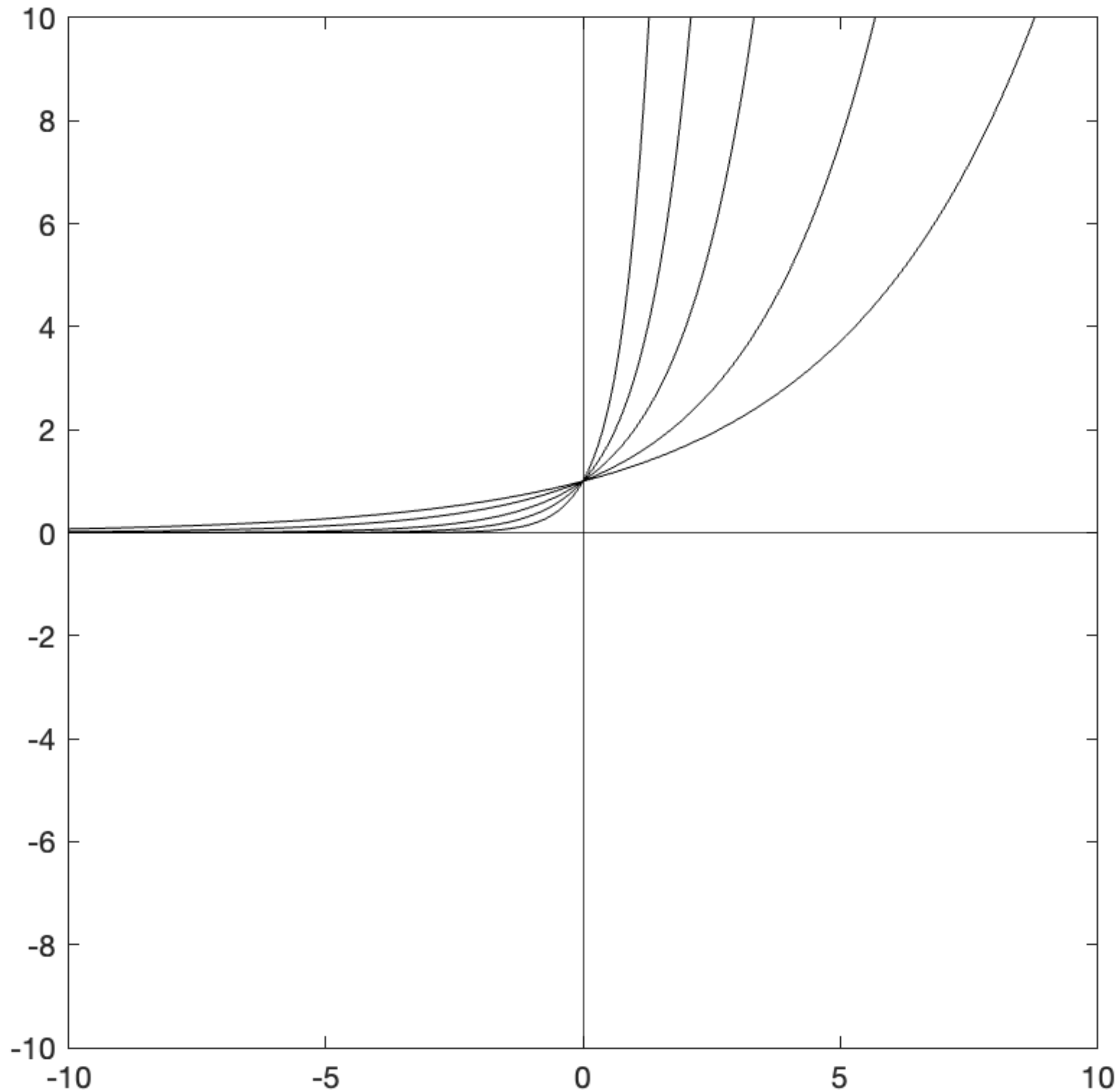
What should our master base be?



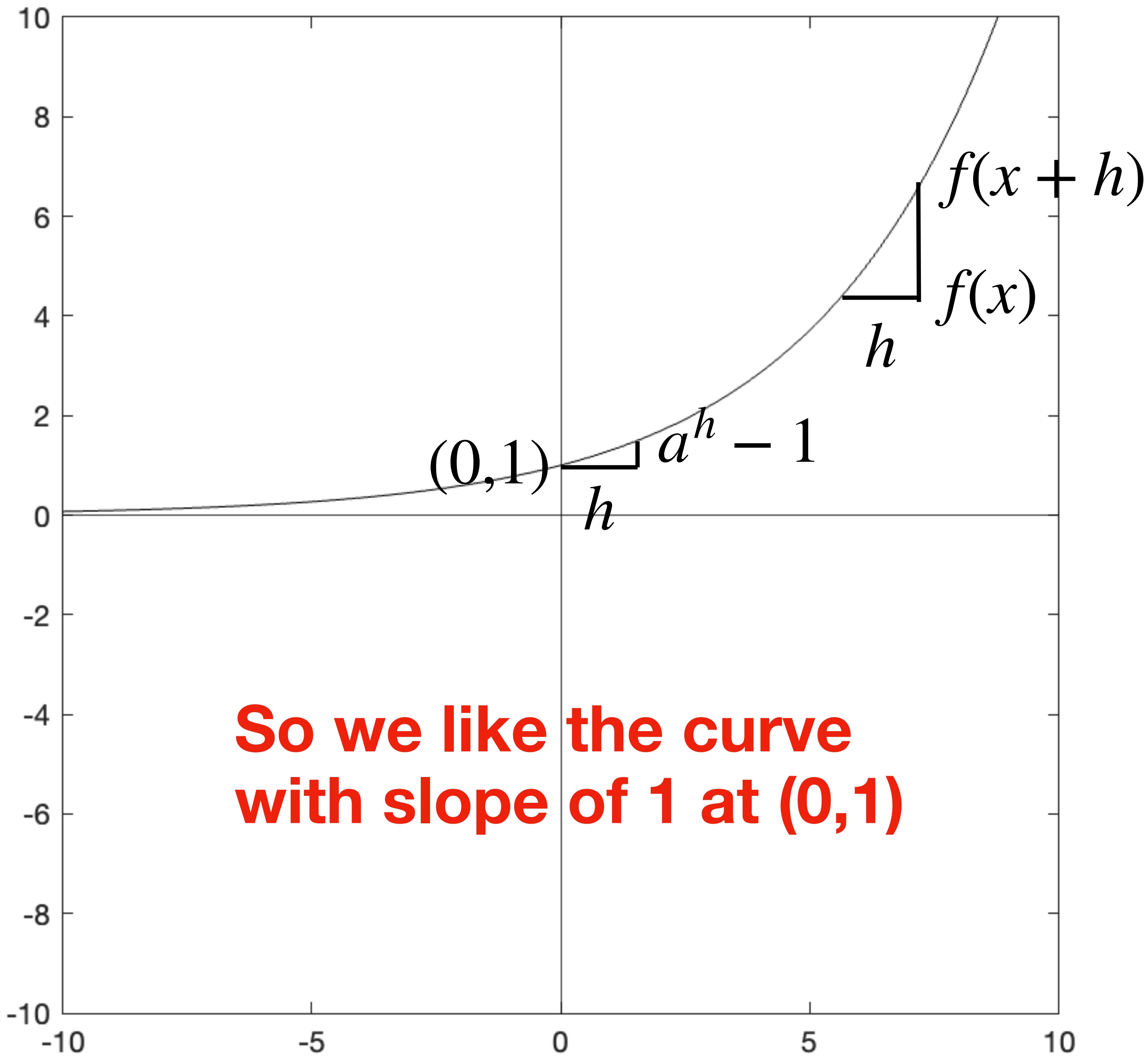


12
10
8
6
4
2
0
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x
$y = \left(\frac{1}{3}\right)^x$
$y = 3^x$

What should our master base be?



What should our master base be?



$$f'(x) : \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) : \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

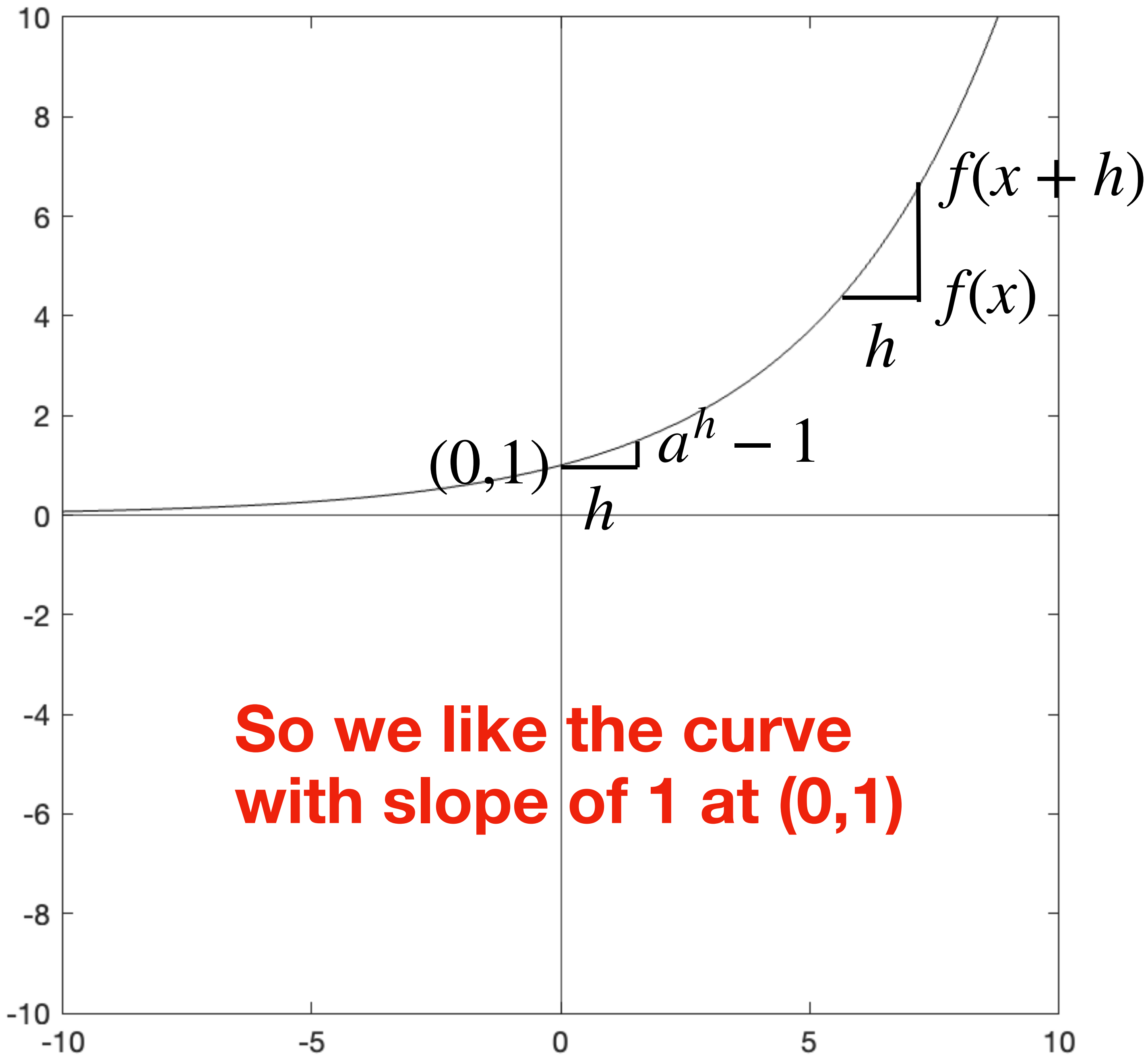
$$f'(x) : \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

$$f'(x) : a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

$$f'(x) = f(x) \times \lim_{h \to 0} \frac{a^h - 1}{h}$$

If we choose base a so this limit = 1, then: $f'(x) = f(x)$

What should our master base be?



$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

$$a^h - 1 = h$$

$$a^h = 1 + h$$

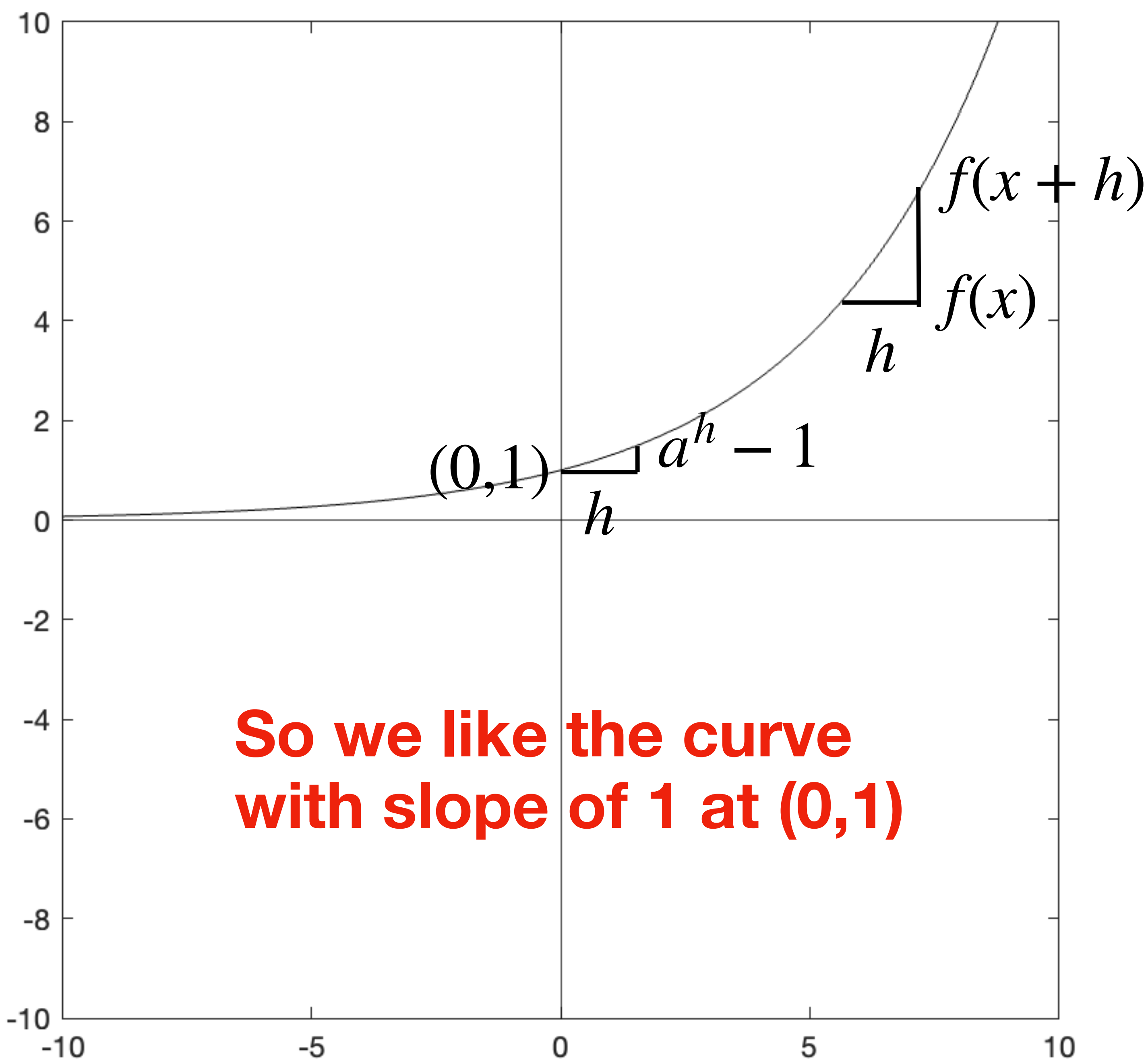
$$a = (1 + h)^{\frac{1}{h}} \qquad n = \frac{1}{h}$$

$$a = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

$$f'(x) = f(x) \times \lim_{h \to 0} \frac{a^h - 1}{h}$$

If we choose base a so this limit = 1, then: $f'(x) = f(x)$

What should our master base be?



$$\lim_{h \to 0} \frac{a^h - 1}{h} = 1$$

$$a^h - 1 = h$$

$$a^h = 1 + h$$

$$a = (1+h)^{\frac{1}{h}} \qquad n = \frac{1}{h}$$

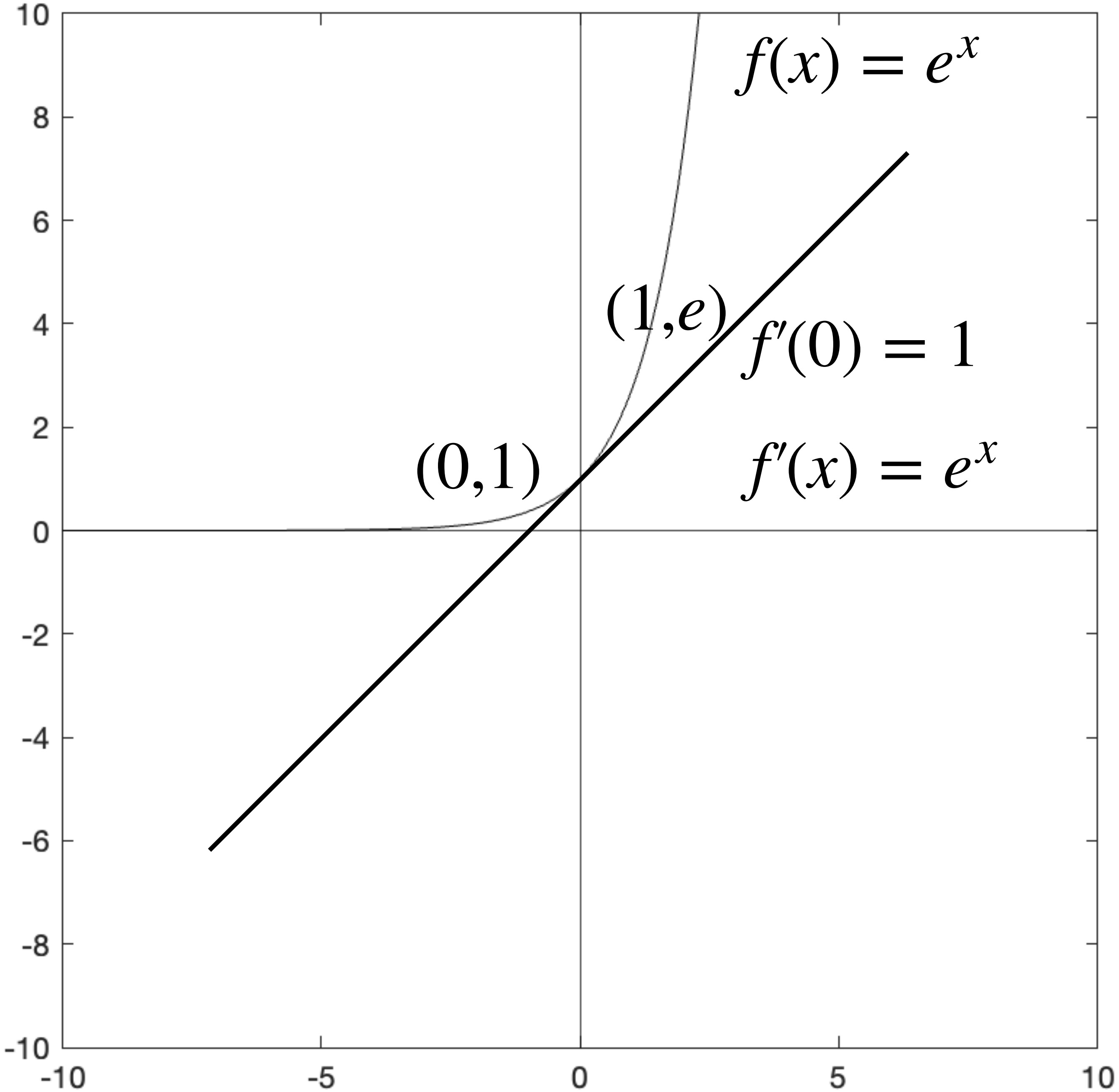
$$a = \lim_{n \to \infty} (1 + \frac{1}{n})^n$$

$$[\frac{2}{1}]^1 \quad [\frac{3}{2}]^2 \quad [\frac{4}{3}]^3 \quad [\frac{5}{4}]^4 \longrightarrow 2.71$$

2   2.25   2.37   2.44

What should our master base be?

"the" exponential function



2.7 1828 1828 45 90 45

e, the Euler number

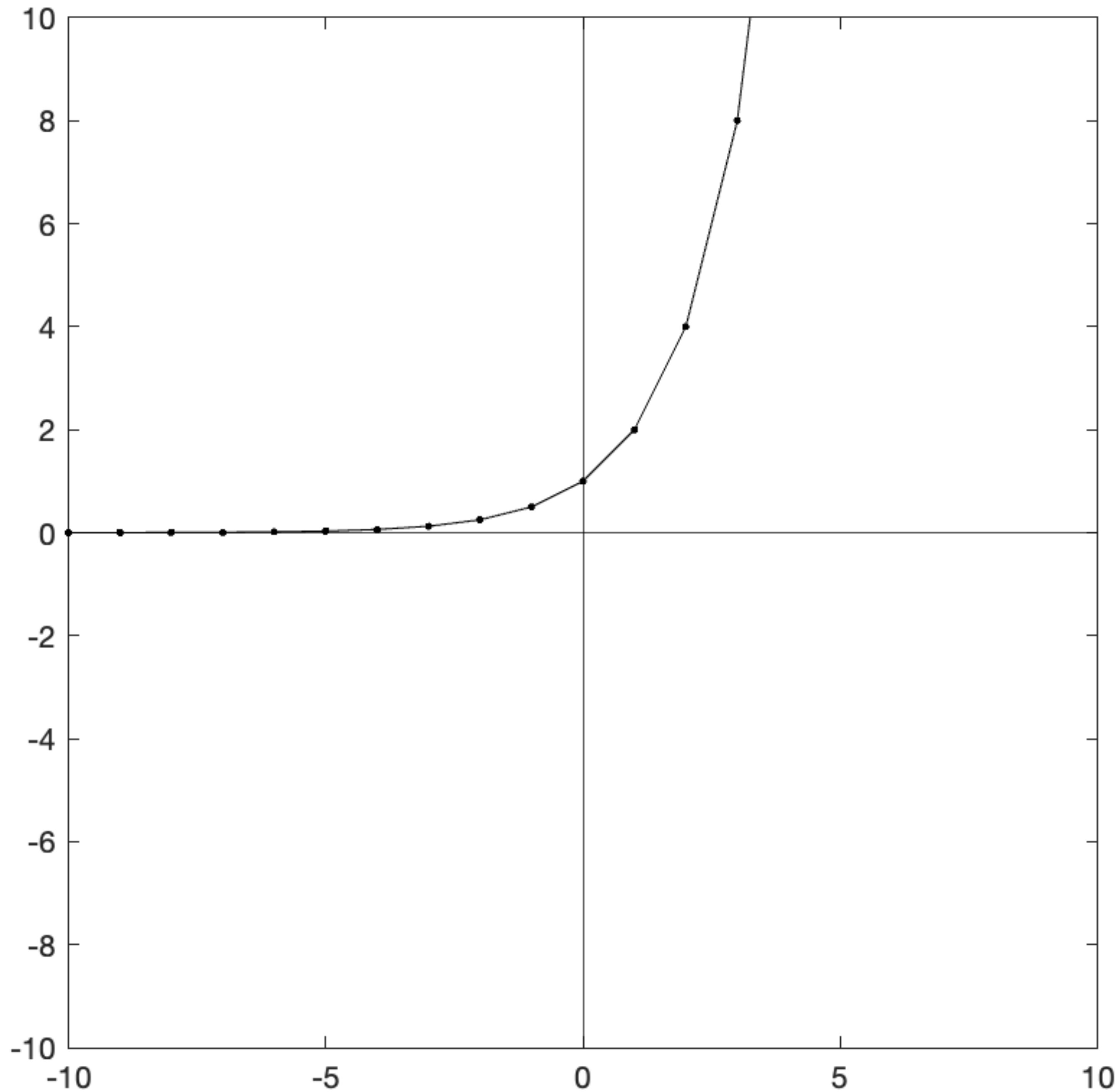
$$e = \lim_{n\to\infty} (1 + \frac{1}{n})^n$$

$$\frac{d}{dx}e^x = e^x$$

$$[\frac{2}{1}]^1 \quad [\frac{3}{2}]^2 \quad [\frac{4}{3}]^3 \quad [\frac{5}{4}]^4 \longrightarrow 2.71$$

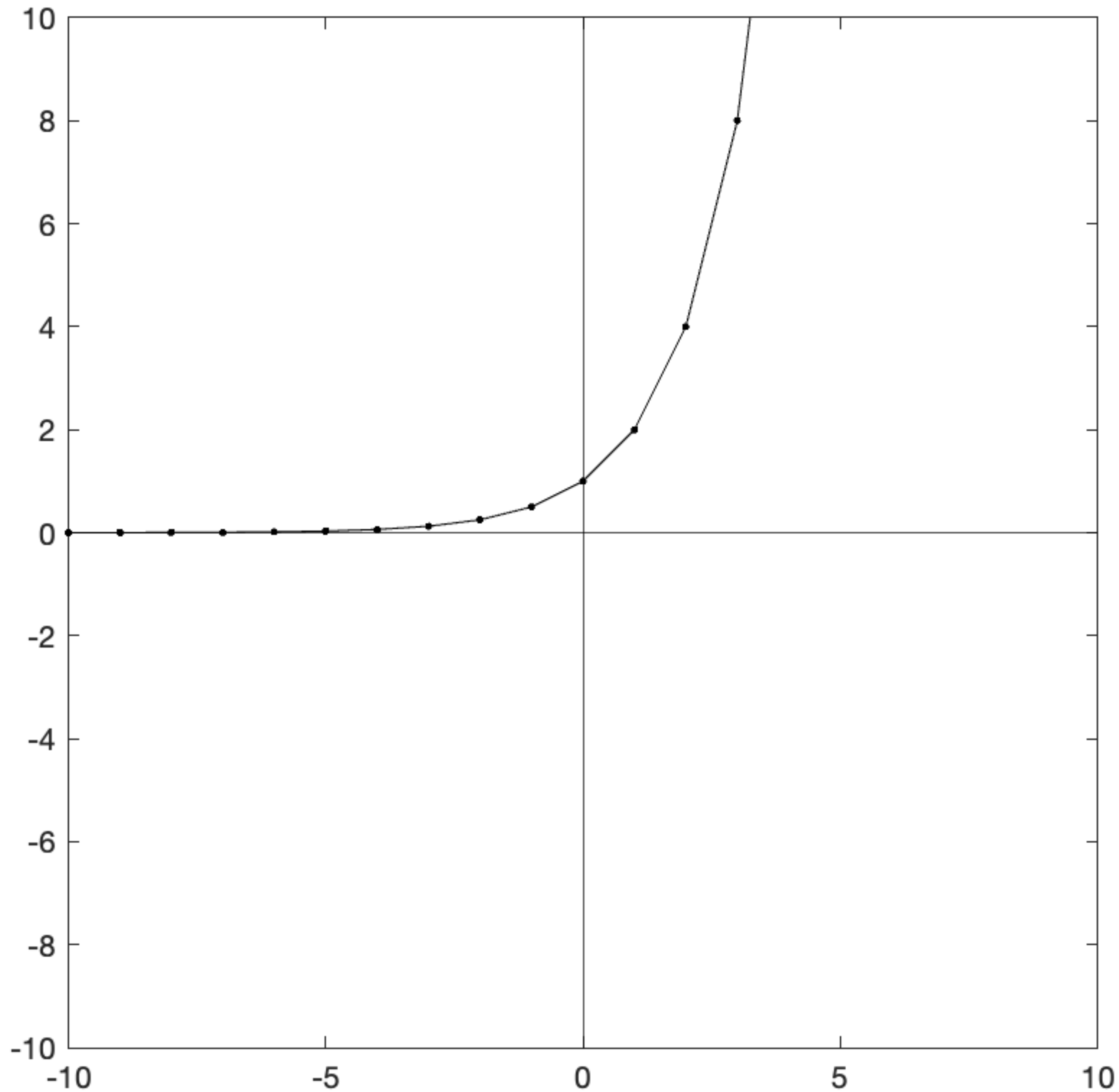
2 2.25 2.37 2.44

Algebraic: 1, 2, 4, 8…



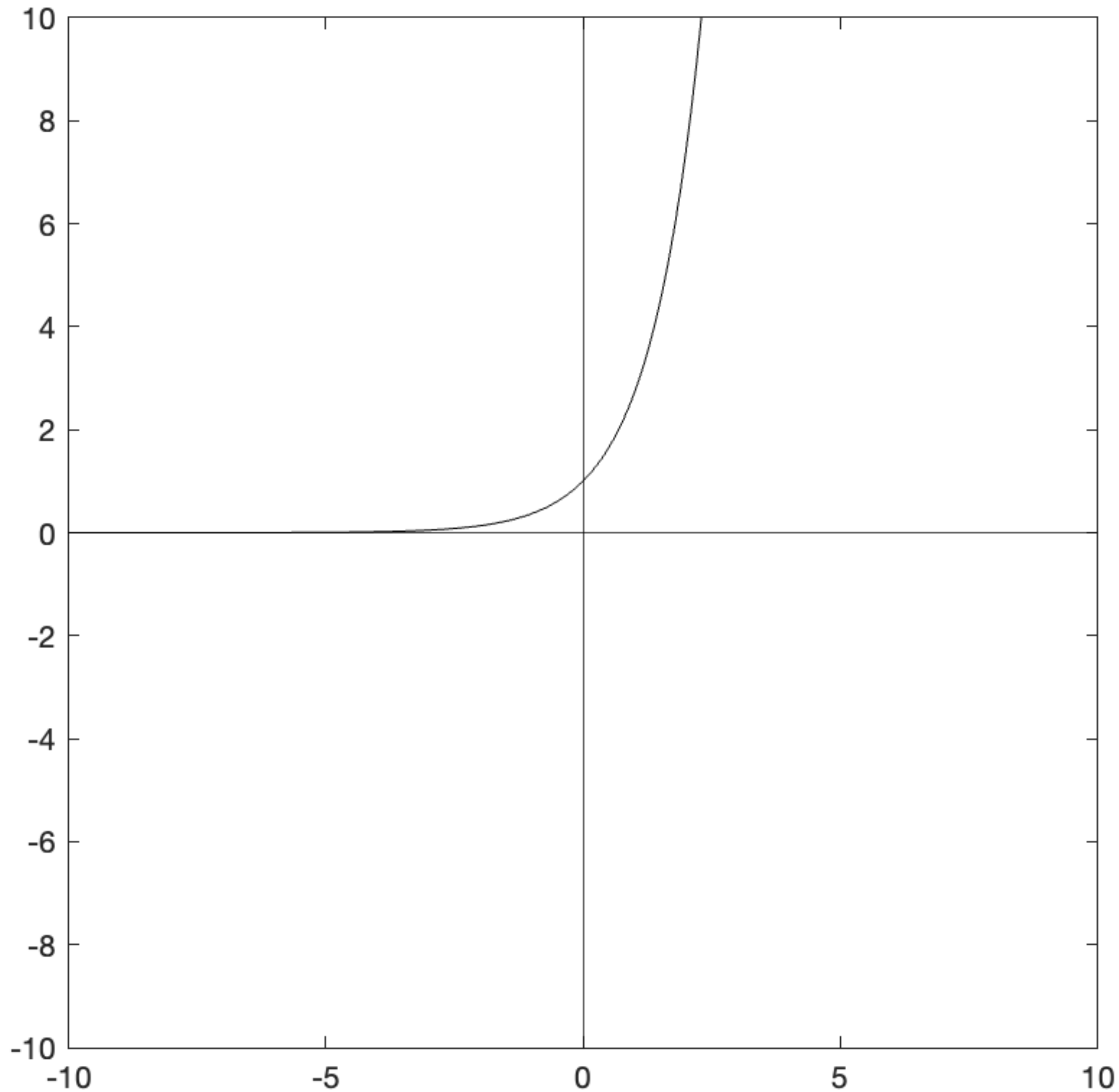
$$y = 2^x$$

Maybe: a process repeated x times - each step doubles

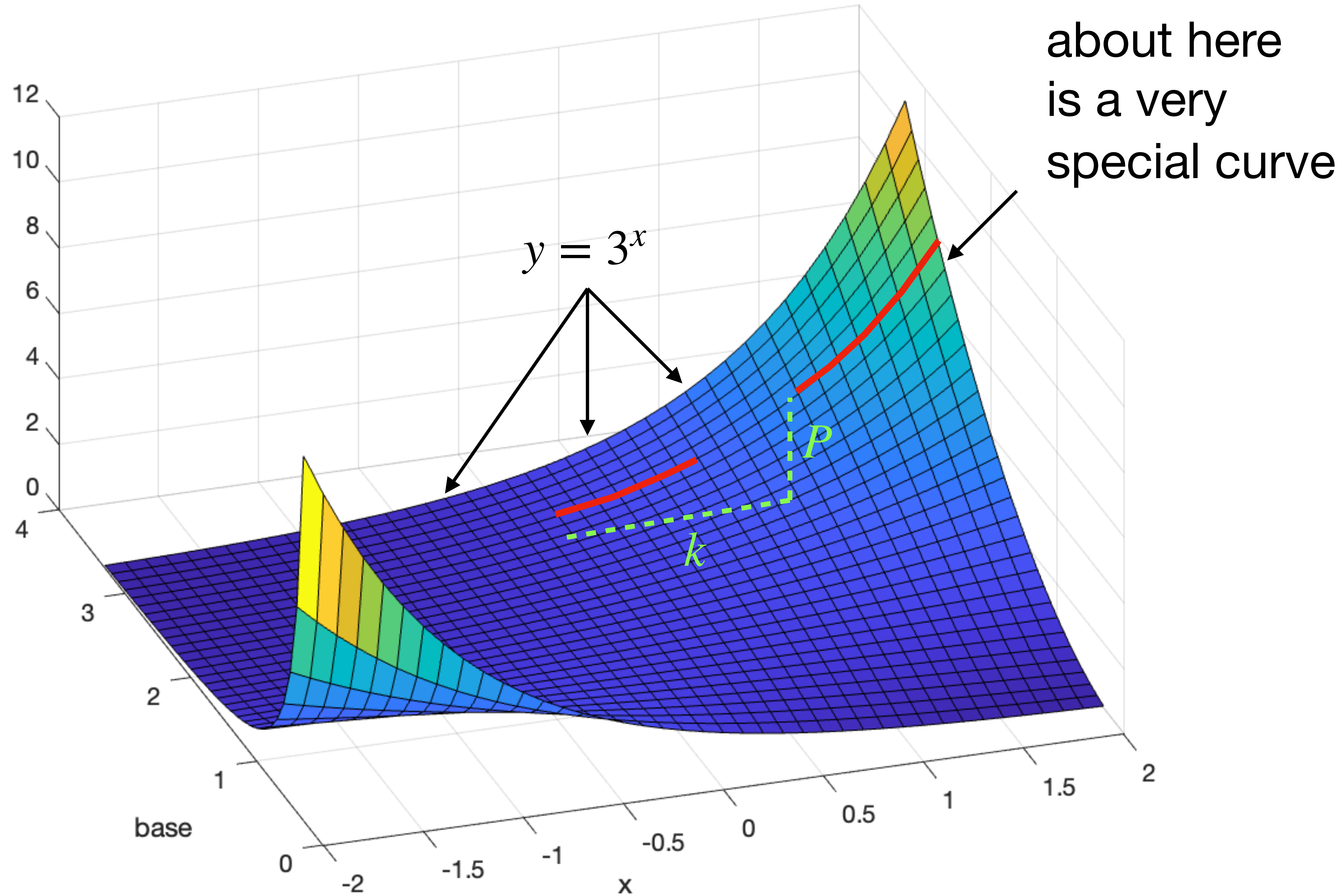


$$y = 2^x$$

Maybe: a process of growth. The rate ***<u>IS</u>*** the function



$$y = e^x$$



about here
is a very
special curve
$y = 3^x$
P
k
12
10
8
6
4
2
0
4
3
2
1
0
base
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
x

All exponential curves are fundamentally a single curve

$$y = a^{bx+c}$$

$a : master\ base \quad (some\ number)$

$b : conversion\ to\ other\ base \quad (stretch\ on\ x)$

$c : shift\ to\ starting\ height \quad (scale\ on\ y)$

THIS COVERS ALL POSSIBLE EXPONENTIAL FUNCTIONS

All exponential curves are fundamentally a single curve

$$y = e^{bx+c}$$

$a : master\ base \quad (some\ number)$

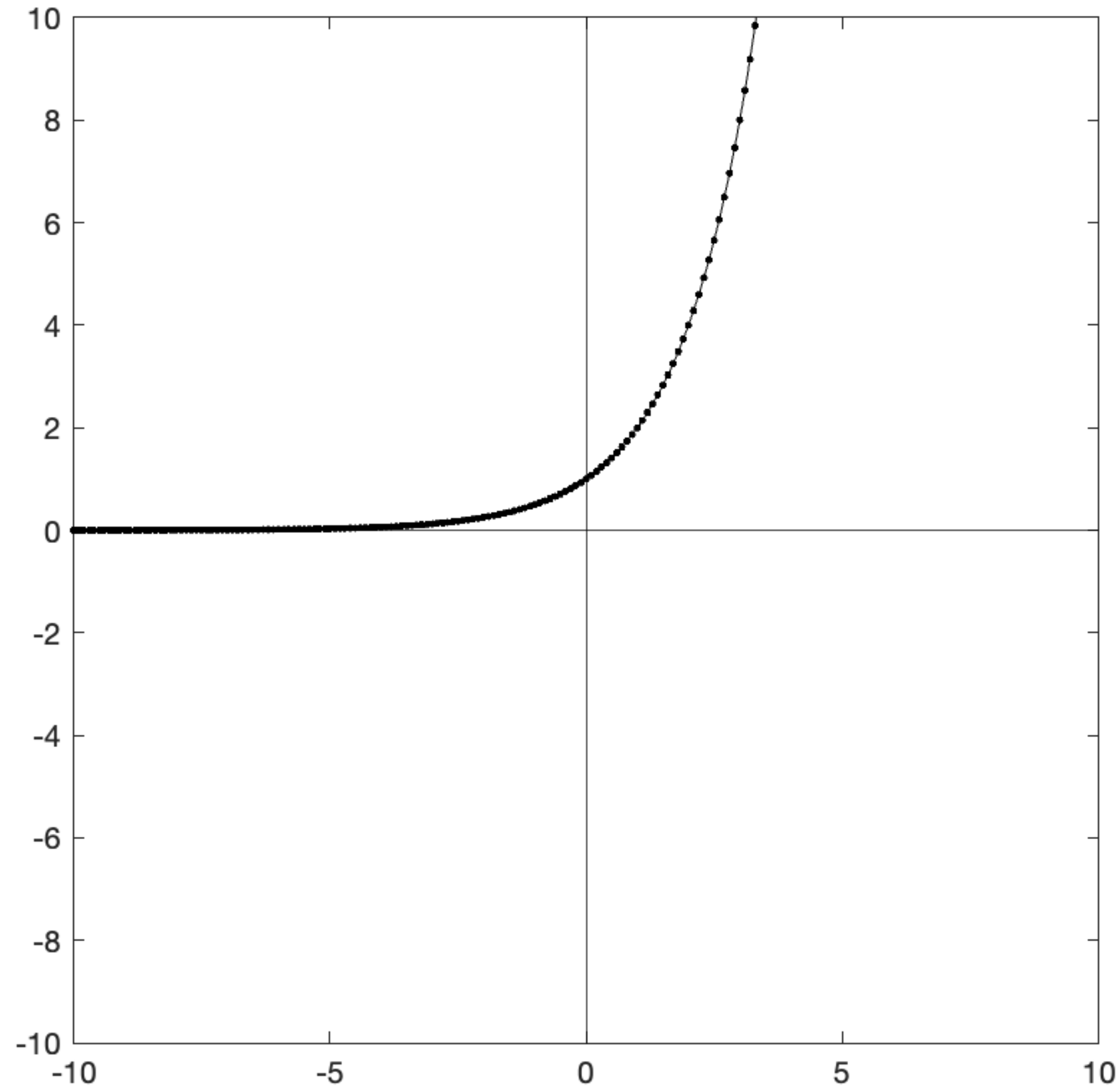
$b : conversion\ to\ other\ base \quad (stretch\ on\ x)$

$c : shift\ to\ starting\ height \quad (scale\ on\ y)$

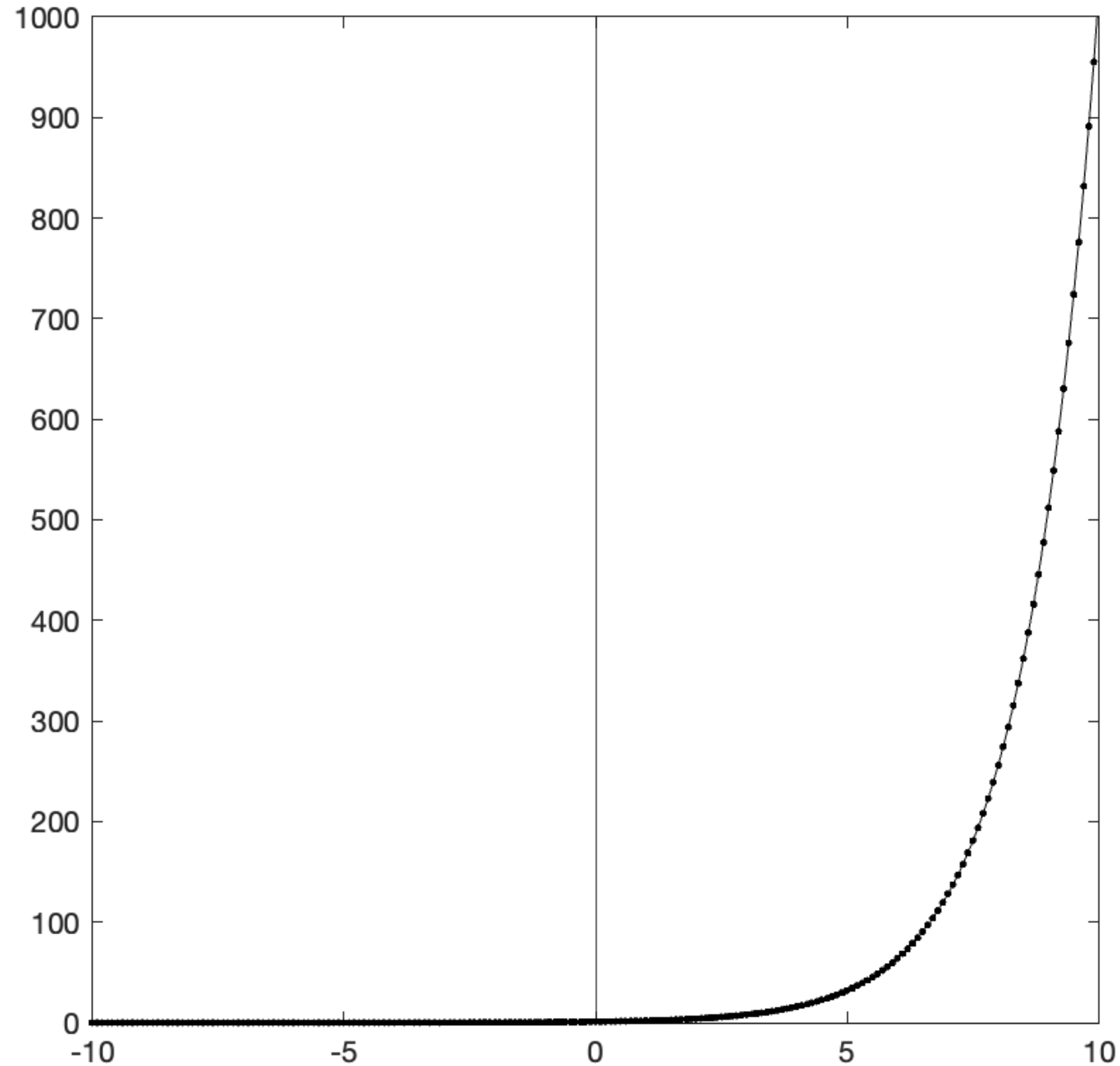
THIS COVERS ALL POSSIBLE EXPONENTIAL FUNCTIONS

A note on seeing exponential functions

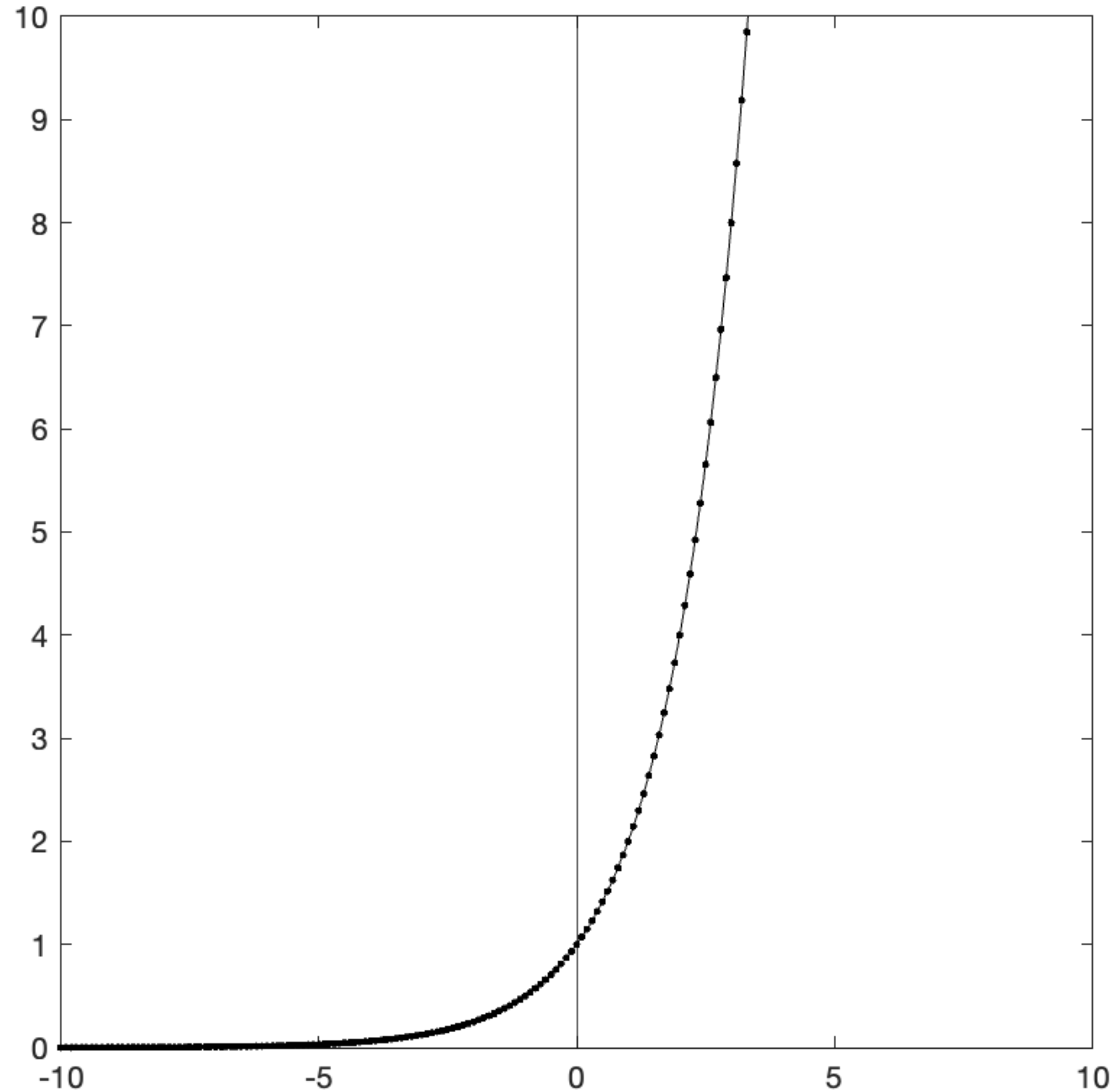
$$y = 2^x$$



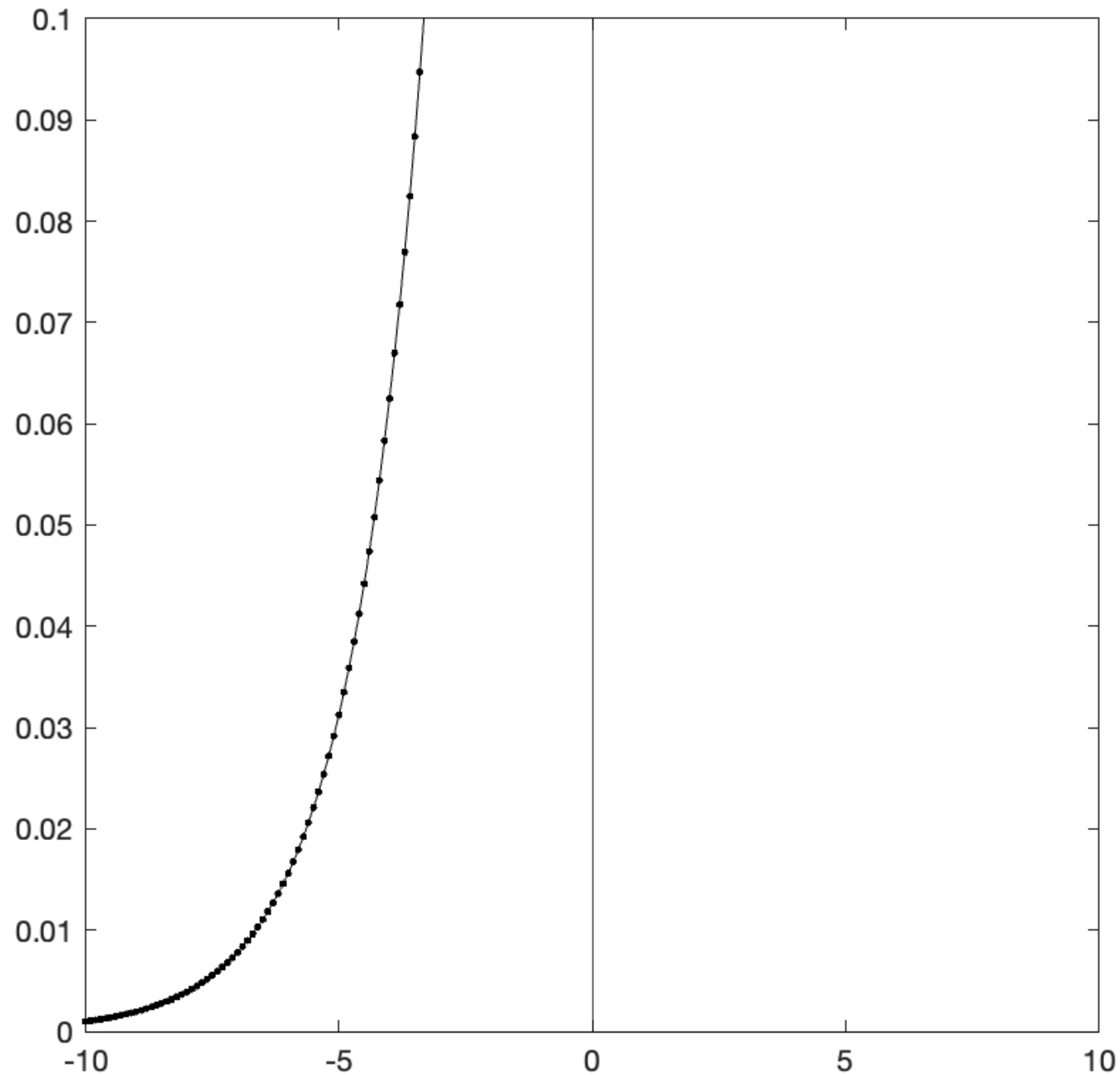
$$y = 2^x$$



$$y = 2^x$$

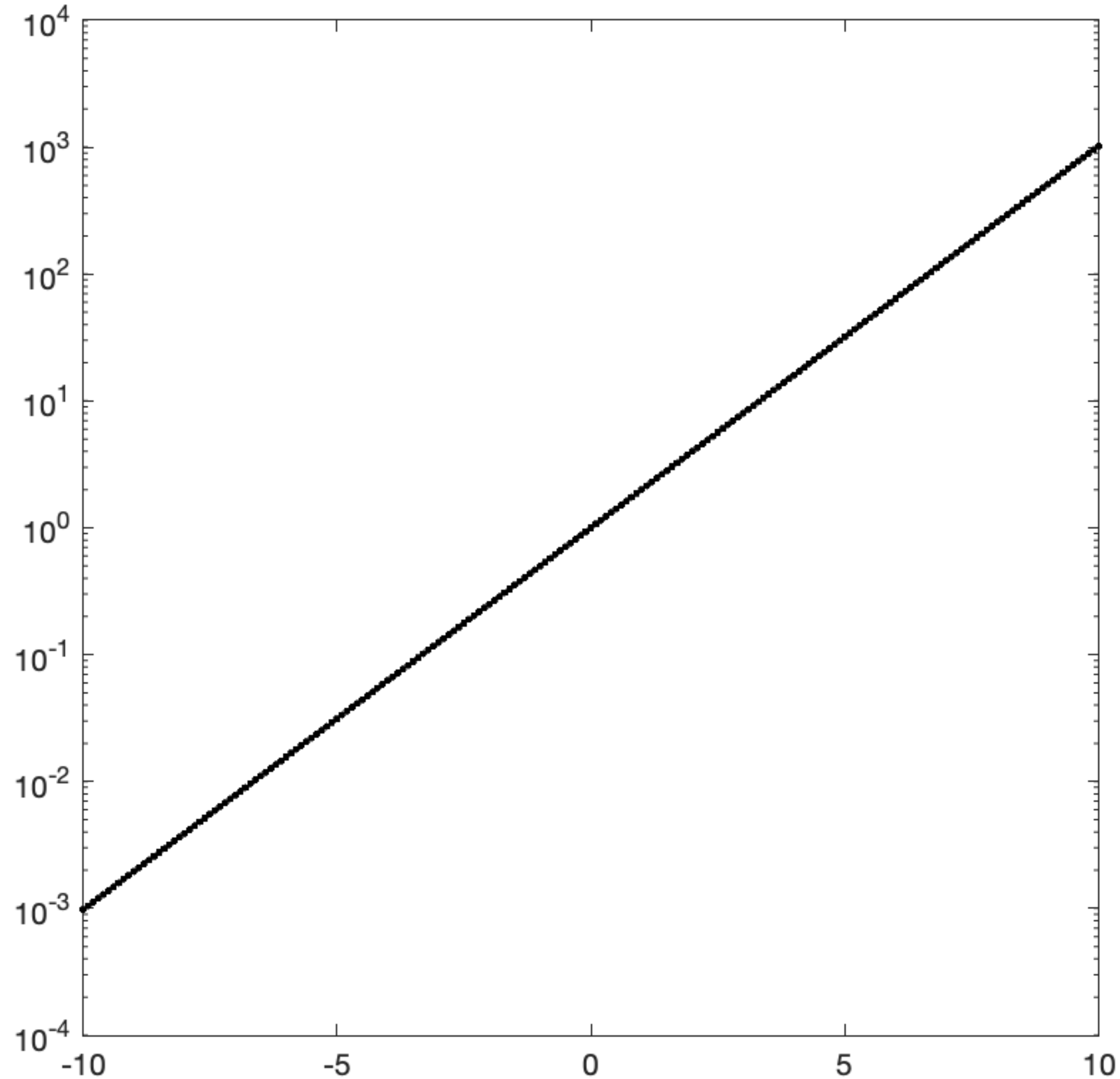


$$y = 2^x$$



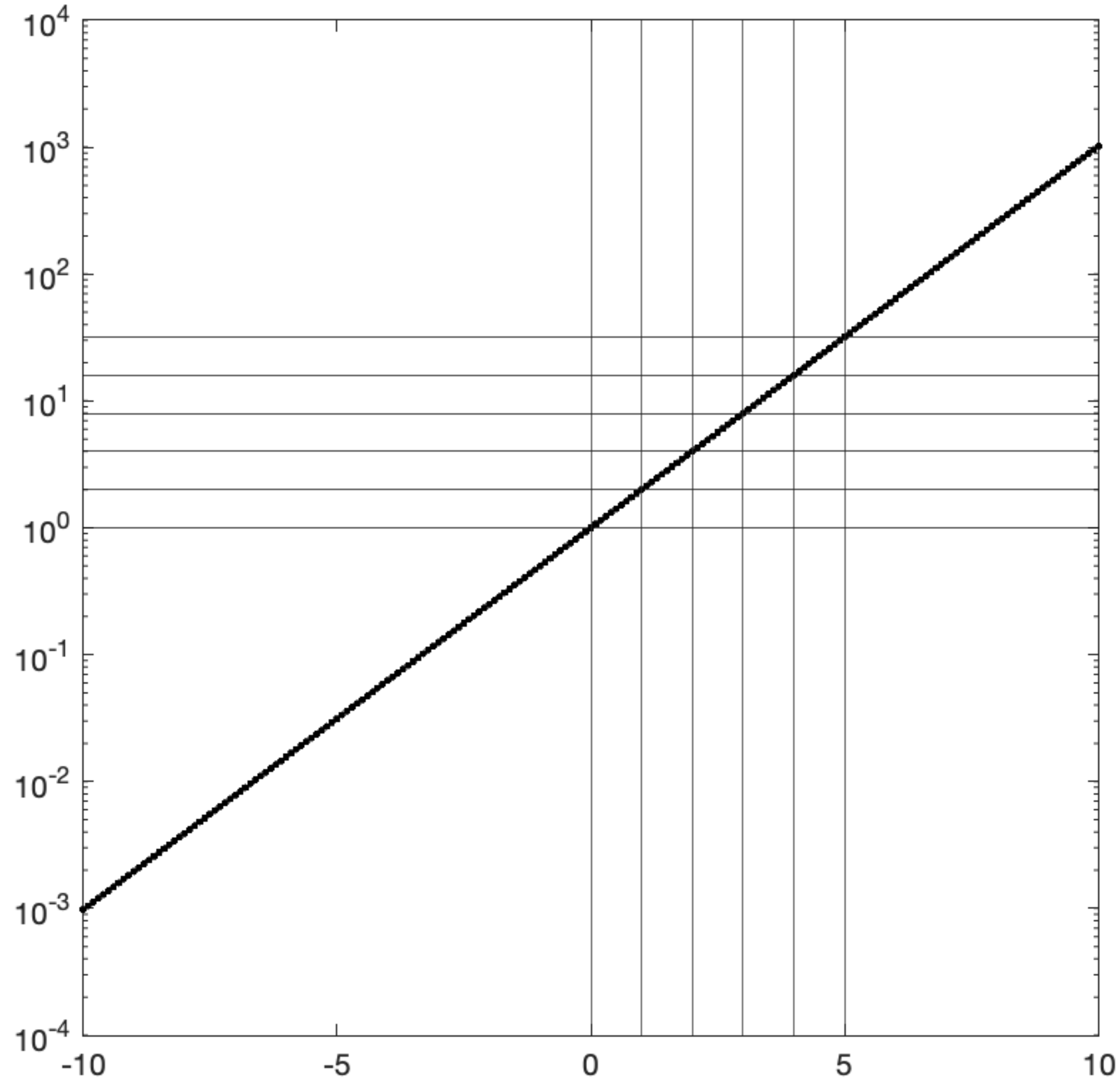
$y = 2^x$

ticks: 10-fold



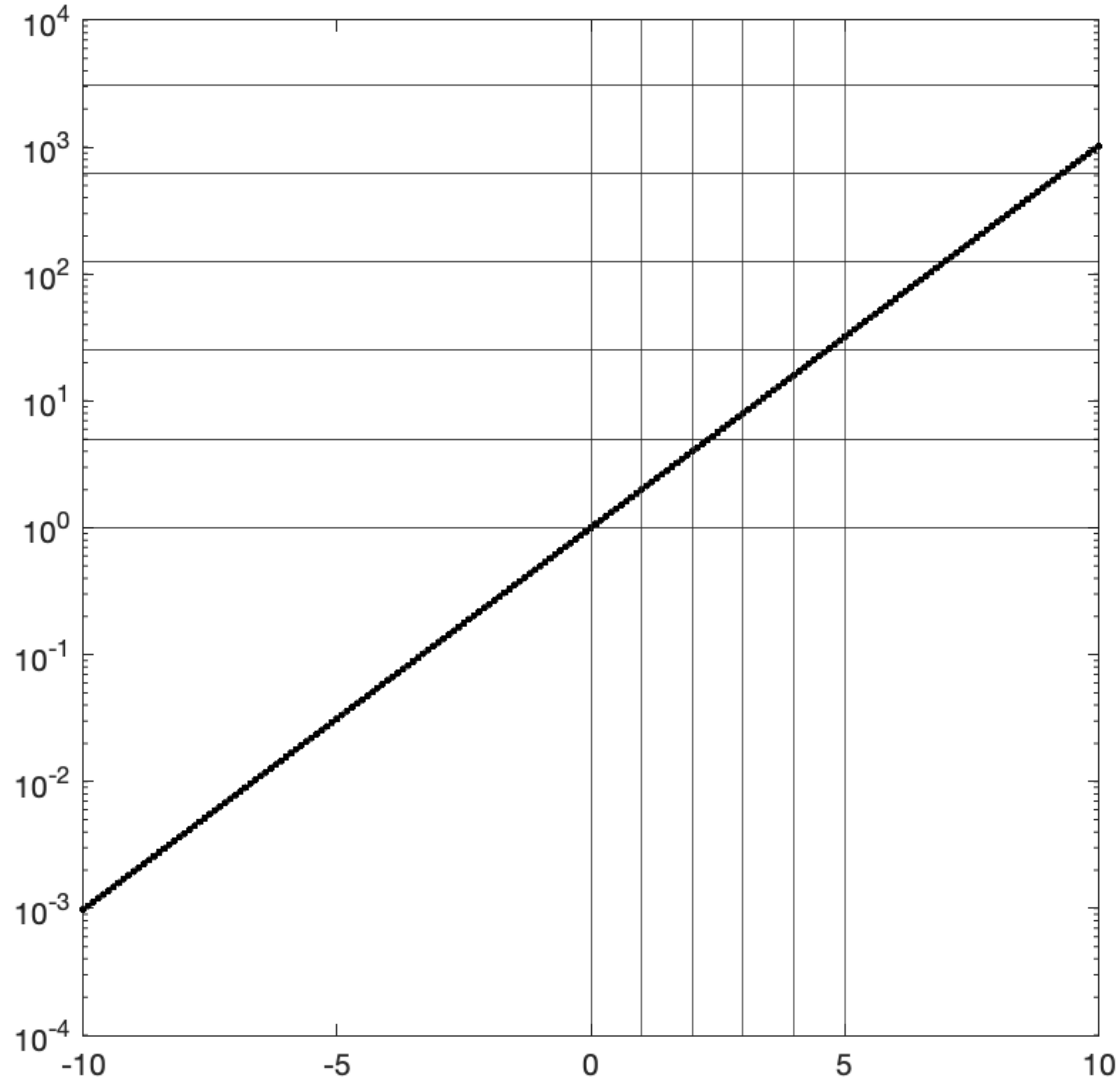
$y = 2^x$

lines: 2-fold



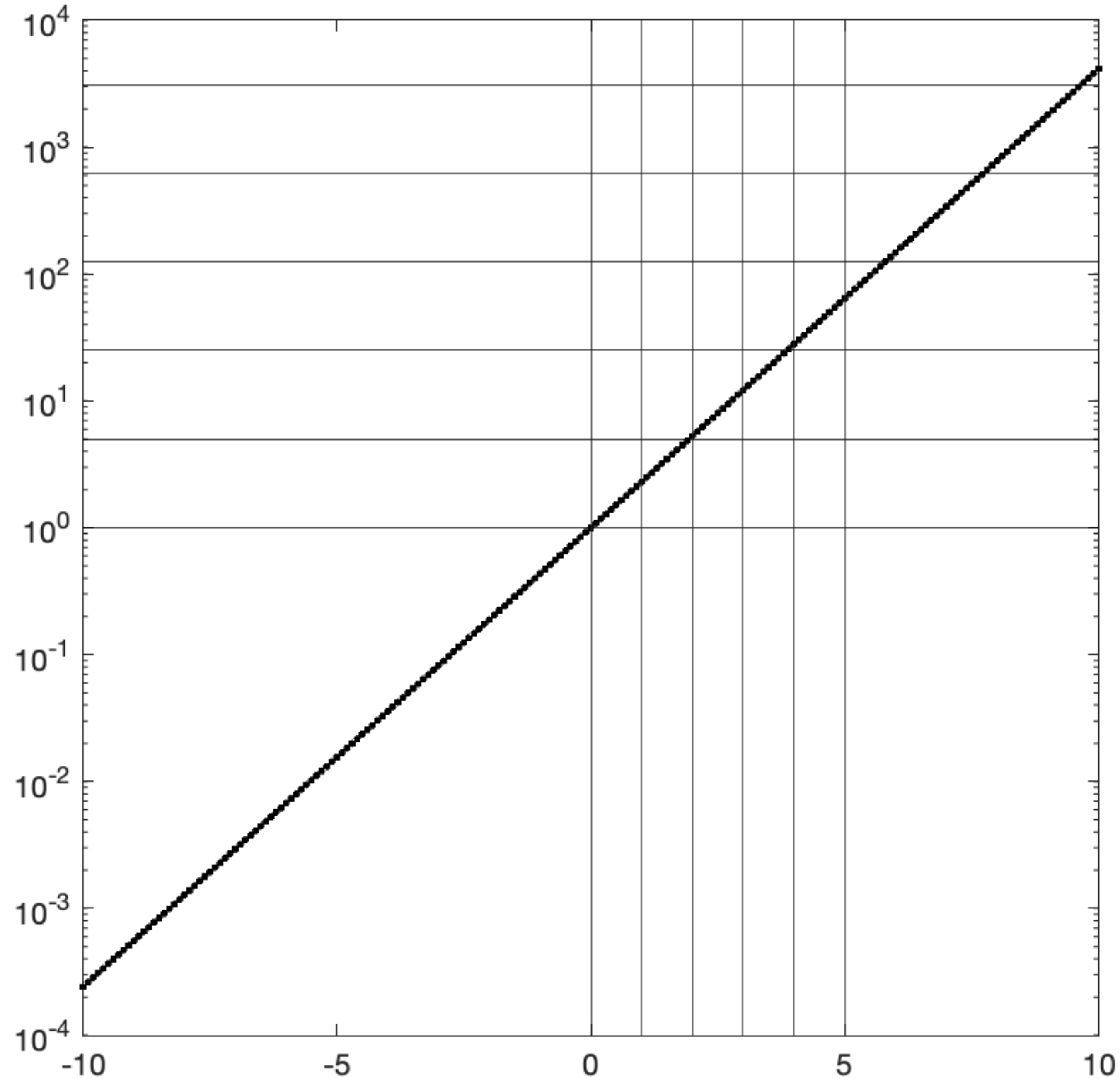
$y = 2^x$

lines: 5-fold



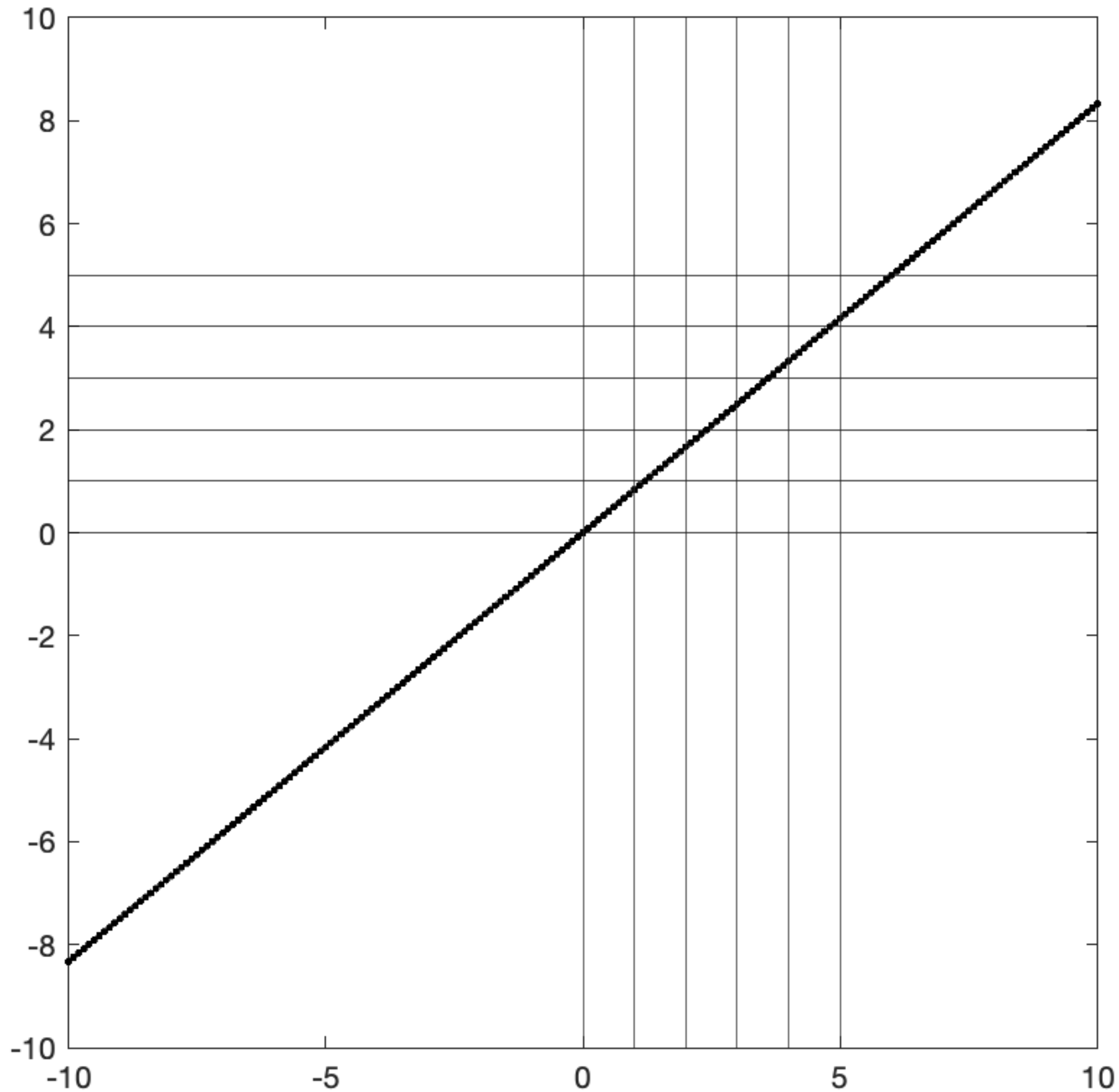
$y = 2.3^x$

lines: 5-fold



$$y = 2.3^x$$

lines: e-fold



A note on seeing exponential functions

Exponentials are impossible to see well in linear plots

So we create plots where "unit y = a multiplicative ratio"

These plots render all exponentials linear, since exponentials are just repeated applications of ratios

The base used for y does not matter, nor does the "native" base of the function, those only alter the slope, not the linearity.

END PART 2